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1	Variational Retrieval of Rain Microphysics and Related Parameters from Polarimetric
2	Radar Data with a Parameterized Operator
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ABSTRACT

A variational retrieval of rain microphysics from polarimetric radar data (PRD) has been developed through the use of S-band parametrized polarimetric observation operators. Polarimetric observations allow for the optimal retrieval of cloud and precipitation microphysics for weather quantification and data assimilation for convective scale numerical weather prediction (NWP) by linking PRD to physical parameters.

Rain polarimetric observation operators for reflectivity (Z_H), differential reflectivity (Z_{DR}), and specific differential phase (K_{DP}) were derived for S-band PRD using T-matrix scattering amplitudes. These observation operators link the PRD to the physical parameters of water content (W) and mass/volume-weighted diameter (D_m) for rain, which can be used to calculate other microphysical information.

40 The S-band observation operators were tested using a 1D variational retrieval that uses the (nonlinear) Gauss-Newton method to iteratively minimize the cost function to find an 41 42 optimal estimate of $D_{\rm m}$ and W separately for each azimuth of radar data, which can be applied to 43 a plan position indicator (PPI) radar scan (i.e., a single elevation). Experiments on Two-44 Dimensional Video Disdrometer (2DVD) data demonstrated the advantages of including $\Phi_{\rm DP}$ 45 observations and using the nonlinear solution rather than the (linear) Optimal Interpolation (OI) solution. PRD collected by the KOUN WSR-88D on 15 June 2011 were used to successfully test 46 the retrieval method on radar data. The successful variational retrieval from the 2DVD and the 47 48 radar data demonstrate the utility of the proposed method.

2

49 1. Introduction

50 Observation-based radar retrievals utilize radar data to retrieve rain microphysics information. Basic retrievals use empirical formulas to calculate rain microphysics information 51 52 on a gate-by-gate basis (e.g., Z-R relationship for rain rate) without accounting for statistics of the observation errors. Advanced techniques such as Optimal Interpolation (OI) (Eliassen 1954; 53 54 Gandin 1963) and variational methods (Lorenc 1986), which have been used for data 55 assimilation (DA) in numerical weather prediction (NWP), can be utilized in observation-based retrievals to account for these statistics. Early studies on radar observation-based retrievals using 56 57 variational methods focused on retrievals of three-dimensional wind fields from Doppler velocity 58 observations (Sun et al. 1991; Qiu and Xu 1992; Laroche and Zawaziki 1994; Gao et al. 1999; Gao et al. 2001). Hogan (2007), Cao et al. (2013), and Yoshikawa et al. (2014) used radar 59 60 hydrometer related observations to optimally retrieve microphysics information by accounting for background and/or observation errors (i.e., spatial covariance is taken into account). 61

A variational method utilizes forward observation operators. A forward observation operator (or forward model) is a transformation, based on physical laws, which converts NWP model state variables to observations (Kalnay 2003). Rodgers (2000) states that "the heart of a successful and accurate retrieval method is the forward model". Throughout the literature, the forward observation operators are simply called observation operators. The best observation operators must include relevant physics, are numerically efficient, and have easily calculated first derivatives (i.e., the Jacobian) (Rodgers 2000).

An observation-based radar retrieval is beneficial if it can be easily used in convective
 scale NWP to improve model microphysics parameterization and weather forecasts. Single
 moment and double moment microphysics parameterization schemes are commonly used in

high-resolution convective scale NWP models (e.g., Lin et al. 1983; Milbrandt and Yau 2005; 72 73 Morrison et al. 2005). In convective-scale NWP model simulations with single-moment microphysics, rain-water mixing ratio (q_r) is the only prognostic variable for hydrometeor 74 physics. In double-moment microphysical parameterization schemes, both q_r and the number 75 concentration (N_t) are predicted. N_t and q_r can be converted to water content ($W = \rho_a q_r$, where ρ_a is 76 air density) and mass/volume-weighted diameter (D_m) , which are related to the drop-size 77 78 distribution (DSD) for rain (i.e., W and D_m can be considered "related parameters" to describe 79 rain microphysics). They are also considered "physical parameters" because they describe 80 physical quantities and more likely to have a Gaussian distribution than non-physical quantities. 81 They can be utilized to calculate other microphysical quantities such as rainfall rate. This 82 motivates us to use parameterized polarimetric forward observation operators that link NWP model state variables and radar observed quantities for an observation-based retrieval. 83 Essentially, the goal is to link the prognostic physics variables of N_t and q_r to the polarimetric 84 85 radar variables by retrieving W and $D_{\rm m}$ for rain.

In this study, an observation-based variational retrieval of rain microphysics from polarimetric radar data (PRD) has been developed through the use of derived parametrized polarimetric observation operators and a nonlinear, iterative solution.

The variational retrieval and derived observation operators are tested by conducting experiments on 2-dimensional video disdrometer (2DVD; Kruger and Krajewski 2002) data and radar data. Using 2DVD data, the intrinsic or true values of the retrieved variables are known and simulated observations can be derived. Since the truth is known, the 2DVD experiments can quantitatively evaluate the variational retrieval and observation operators for different conditions. These experiments will be discussed in context of the both the retrieval of *W* and *D*_m 95 (i.e., the inverse problem) and the estimation of polarimetric radar variables (i.e., the forward
96 problem) to test the observation operators. In other words, the observation operators are tested
97 both going backward from the observations to the state variables and forward from the state
98 variables to observations.

99 Once the variational retrieval and observation operators are successfully tested on 2DVD 100 data and simulated observations, the next step is to apply them to real radar data. The goal of the 101 radar data experiments is to demonstrate the variational retrieval on a both a single azimuth and 102 an entire elevation scan of PRD.

103 This manuscript is organized as follows. The derivation of the parameterized polarimetric 104 observation operators is shown in section 2. This is followed by a description of the variational 105 retrieval methodology in section 3. Next, the polarimetric observation operators and variational 106 retrieval methodology are applied to DSD data collected by a disdrometer in section 4 and to 107 radar data in section 5. A summary of results, conclusions, and future work for this study are 108 provided in the final section.

109

110 2. Parameterized Polarimetric Radar Forward Observation Operators

The DSD is the fundamental description of rain microphysics (Zhang 2016). Therefore, a DSD model will be utilized to derive the observation operators. The DSD is defined as the number of drops in a unit volume for each unit diameter bin. It is a function of equivalent drop diameter, D, and is expressed as expressed as N(D) (# m⁻³ mm⁻¹). D is the diameter of a sphere that has an equivalent volume to an oblate spheroid. The raindrops are assumed to be spheroids that become more oblate as their size increases. In this study, the two-parameter exponential distribution will be used for the derivation of the observation operators:

$$N(D) = N_0 \exp(-\Lambda D) \tag{1}$$

119 where N_0 (m⁻³ mm⁻¹) is the intercept parameter and Λ (mm⁻¹) is the slope parameter. These are 120 often called "DSD parameters".

For a two-parameter DSD model, the model physics parameters N_t and q_r can be converted to W and D_m . Therefore, the state variables chosen for the observation operators are Wand D_m . Note that W and D_m can be derived using existing DSD retrievals that retrieve N_0 and Λ ; however, the advantage is these are physical parameters, which tend to be Gaussian distributed. Non-physical parameters are less likely to have a Gaussian distribution. N_0 and Λ do not have a Gaussian distribution (e.g., Cao et al. 2010; Zhang 2016). Cao et al. (2010) showed that N_0 and Λ have greatly skewed distributions.

128 The first state variable, *W* [g m⁻³], is derived using the definition of the volume of a 129 sphere ($V = D^3 \pi/6$) and the third DSD moment:

130
$$W = \frac{\pi}{6} \rho_w \int_0^{D_{max}} D^3 N_0 \exp(-\Lambda D) dD \qquad (2a)$$

131
$$= \frac{\pi}{6} \times 10^{-3} N_0 \Lambda^{-4} \ \gamma(\Lambda D_{max}, 4)$$
(2b)

132 where $\rho_{\rm w}$ is the density of water and γ is the incomplete gamma function.

133 As previously noted, W is directly related to q_r as $W = \rho_a q_r$.

134 The second state variable, $D_{\rm m}$ [mm], is derived using the ratio of the fourth and third 135 DSD moments:

136
$$D_m = \frac{\int_0^{D_{max}} D^4 N_0 \exp(-\Lambda D) dD}{\int_0^{D_{max}} D^3 N_0 \exp(-\Lambda D) dD}$$
(3*a*)

137
$$= \frac{1}{\Lambda} \quad \frac{\gamma(\Lambda D_{max}, 5)}{\gamma(\Lambda D_{max}, 4)}$$
(3b)

138

139 N_t is related to N_0 by the zeroth DSD moment:

140
$$N_t = \int_0^{D_{max}} N_0 \exp(-\Lambda D) dD$$
(4*a*)

141
$$= \frac{N_0}{\Lambda} \gamma(\Lambda D_{max}, 1)$$
(4b)

142 The relationship of $D_{\rm m}$ to $q_{\rm r}$ and $N_{\rm t}$ can be derived by first solving for N_0 from (2) and Λ 143 from (4). By substituting N_0 into the equation for Λ , the following equation for Λ is found:

144
$$\Lambda = \left(\frac{\frac{\pi}{6}\rho_w N_t \gamma(\Lambda D_{max}, 4)}{\rho_a q_r \gamma(\Lambda D_{max}, 1)}\right)^{1/3}$$
(5)

145 Therefore, using the definition of $D_{\rm m}$ from (3), $D_{\rm m}$ is related to $q_{\rm r}$ and $N_{\rm t}$ by the following 146 relationship:

147
$$D_m = \frac{\gamma(\Lambda D_{max}, 5)}{\gamma(\Lambda D_{max}, 4)} \left(\frac{\rho_a q_r \gamma(\Lambda D_{max}, 1)}{\frac{\pi}{6} \rho_w N_t \gamma(\Lambda D_{max}, 4)} \right)^{\frac{1}{3}}$$
(6)

148 The dual-polarized radar variables used for the derivation of the observation operators are 149 the horizontal radar reflectivity factor (Z_h), differential reflectivity ($Z_{DR}=10\log_{10}(Z_{dr})$, where Z_{dr} 150 is in linear units), specific differential phase (K_{DP}), and copolar correlation coefficient (ρ_{hv}).

151 The $K_{\rm DP}$ observation operator will be applied through the use of $\Phi_{\rm DP}$. $\Phi_{\rm DP}$ is the total 152 differential phase, which is the sum of the differential backscattering phase and differential 153 propagation phase (or differential phase). It is assumed that contributions of differential 154 backscattering phase are negligible at S-band for rain. With the assumption that differential 155 backscattering phase is negligible, $\Phi_{\rm DP}$ is assumed to only be an estimate of the differential 156 propagation phase, which is the integral of the specific differential phase over the propagation 157 path. Therefore, $\Phi_{\rm DP}$ is directly related to $K_{\rm DP}$ by the following:

$$\phi_{\rm DP}(r) = 2 \int_0^r K_{\rm DP}(l) dl \tag{7}$$

159 $\Phi_{\rm DP}$ is used in the variational retrieval instead of $K_{\rm DP}$ because it is a direct quantity that is 160 measured by radars.

161 The derivation of the observation operators is a three-step process. The first step is to 162 calculate N(D) for a range of D_m using a normalized W of 1 g m⁻³ and assuming an exponential 163 distribution (i.e., Z_h and K_{DP} are normalized so that W is 1 g m⁻³ and D_m only varies). To 164 calculate N(D), Λ is first calculated for a range of D_m using (3). Using Λ , N_0 can be calculated 165 from (2). Once Λ and N_0 are known, N(D) can be calculated using (1). For this derivation, D_m 166 ranges from ~0.08 to ~4.35 mm when using a finite D_{max} of 8 mm.

167 The second step is to calculate Z_h , Z_{dr} , K_{DP} , and ρ_{hv} by using the S-band scattering amplitudes calculated with the T-matrix method and using the N(D) values from the first step. 168 169 The T-matrix method (Waterman 1971) is a numerical-scattering solution. For the scattering 170 amplitude calculations, it is assumed that raindrops will fall with their major axis aligned 171 horizontally (i.e., canting angle mean and standard deviations are zero). It has been found that the true standard deviation for the canting angles for raindrops is somewhere between 0° to 10° 172 173 (e.g., Bringi and Chandrasekar 2001; Ryzhkov et al. 2002). Ryzhkov et al. (2002) found the 174 assumption of no canting angle only results in a slight overestimation (< 6%) in some PRD, so 175 this is a reasonable assumption. The axis ratios of the raindrops assume the following 176 relationship (Brandes et al. 2002):

177

158

$$r = 0.9551 + 0.0251D - 0.03644D^2 + 0.005303D^3 - 0.002492D^4$$
(8)

178 where *r* is the ratio of the semiminor and semimajor axes.

179 The third step is to use a polynomial function to fit the calculated Z_h , Z_{dr} , K_{DP} , and ρ_{hv} to 180 D_m . Because a normalized *W* of 1 g kg⁻¹ was used in the calculation, *W* will be multiplied by the 181 polynomial for Z_h and K_{DP} . These will cancel for Z_{DR} and ρ_{hv} because they are ratios.

182 As a result, here are the derived parametrized polarimetric radar forward observation183 operators for rain:

184
$$Z_{\rm h} \approx W \left(-0.3078 + 20.87 D_m + 46.04 D_{\rm m}^2 - 6.403 D_{\rm m}^3 + 0.2248 D_{\rm m}^4 \right)^2$$
 (9)

185
$$Z_{\rm dr} \approx 1.019 - 0.1430 D_m + 0.3165 D_m^2 - 0.06498 D_m^3 + 0.004163 D_m^4$$
 (10)

186
$$K_{\rm DP} \approx W (0.009260 - 0.08699D_m + 0.1994D_m^2 - 0.02824D_m^3 + 0.001772D_m^4)$$
 (11)

187
$$\rho_{\rm hv} \approx 0.9987 + 0.008289 D_m - 0.01160 D_m^2 + 0.003513 D_m^3 - 0.0003187 D_m^4$$
 (12)

188 or in logarithmic units for $Z_{\rm H}$ and $Z_{\rm DR}$:

189
$$Z_{\rm H} \approx 10\log_{10} \left[W \left(-0.3078 + 20.87D_m + 46.04D_m^2 - 6.403D_m^3 + 0.2248D_m^4 \right)^2 \right]$$
 (13)

190
$$Z_{\rm DR} \approx 10\log_{10}(1.019 - 0.1430D_m + 0.3165D_m^2 - 0.06498D_m^3 + 0.004163D_m^4)$$
 (14)

Figure 1 plots all the observation operators from (9)-(14). These observation operators include relevant physics, are numerically efficient (i.e., polynomials), and are able to easily calculate the first derivative. Note that these calculated polarimetric variables are intrinsic values that do not consider propagation and error effects (e.g., attenuation and sampling errors) and are only valid for S-band radar data.

196

197 **3. Variational Retrieval**

198 The solution to an observation-based variational retrieval is an optimal analysis field, x_a,
199 that minimizes a cost function (Lorenc 1986):

200
$$J(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_b)^T \mathbf{B}^{-1} (\mathbf{x} - \mathbf{x}_b) + [\mathbf{y} - H(\mathbf{x})]^T \mathbf{R}^{-1} [\mathbf{y} - H(\mathbf{x})]$$
(15)

The convention here follows vector-matrix form, which is the notation used by the data assimilation community (e.g., Ide 1997; Huang 2000; Kalnay 2003). The uppercase bold letters represent matrices and the lowercase bold represent vectors in this convention.

The cost function (*J*) is the distance between the analysis (**x**) and the background (**x**_b) weighted by the inverse of the background error covariance (**B**) plus the distance of the analysis (**x**) to the observations (**y**) weighted by the inverse of the observation error covariance (**R**) (Kalnay 2003). *H* is the observation operator. **x** is an analysis vector that is a concatenate of the state variables, *W* and D_m (i.e., the variational retrieval finds an optimal analysis of *W* and D_m). The other terms in the cost function (**x**_b, **B**, **y**, and **R**) are defined in greater detail later in this section.

211 The solution for an observation-based OI retrieval is (Kalnay 2003):

212
$$\mathbf{x}_a = \mathbf{x}_b + \mathbf{W}[\mathbf{y} - H(\mathbf{x}_b)]$$
(16*a*)

213
$$\mathbf{W} = \mathbf{B}\mathbf{H}^{T}(\mathbf{R} + \mathbf{H}\mathbf{B}\mathbf{H}^{T})^{-1}$$
(16*b*)

where **W** is the optimal weight matrix and **H** is the Jacobian, which is the linear approximation of *H*. **H** is derived from equations (11), (13), and (14), which will be shown later in this section.

The assumption with the OI solution is that background is a reasonable approximation of the true analysis, so that the solution of the analysis is equal to the background values plus small increments (Kalnay 2003). The OI solution is usually not solved iteratively (i.e., it is solved explicitly), and is equivalent to the three-dimensional variational assimilation (3DVAR) solution that is found by minimizing the cost function (15).

For moderately nonlinear problems, the Gauss-Newton iterative method can find thesolution (Rodgers 2000):

223
$$\mathbf{x}_{i+1} = \mathbf{x}_b + \mathbf{W}[\mathbf{y} - H(\mathbf{x}_i) + \mathbf{H}_i(\mathbf{x}_i - \mathbf{x}_b)]$$
(17)

$$\mathbf{W} = \mathbf{B}\mathbf{H}_{i}^{T}(\mathbf{R} + \mathbf{H}_{i}\mathbf{B}\mathbf{H}_{i}^{T})^{-1}$$
(17*b*)

225 where \mathbf{H}_i and $H(\mathbf{x}_i)$ are updated during each iteration, *i*.

The Gauss-Newton iterative method is utilized to solve nonlinear least-squares problems (e.g., variational data-assimilation problems) and has been shown to correspond to the incremental four-dimensional variational (4DVAR) assimilation solution (Lawless et al. 2005a, b). For this study, the Gauss-Newton iterative method (17) was used to better account for the nonlinearity in the forward operator.

Note that for any variational retrieval, there is an assumption that the background and observation errors have a Gaussian probability distribution, the background and observation errors are uncorrelated, and the background and observations are unbiased (Kalnay 2003). Errors will be introduced in the variational retrieval if these conditions are not met.

A comparison to the OI solution (16) is also shown in Section 4. The comparison between both methods is shown because even though most retrieval problems are not truly linear, many still utilize linearization about some prior state even when the observation operator is not truly linear (Rodgers 2000). Therefore, the OI solution is tested because it provides a baseline to compare to Gauss-Newton iterative solution and to see if linearity can be assumed.

To solve an observation-based variational retrieval, the state variables and observations must be defined. As previously noted, the state variables for this observation-based retrieval are *W* and $D_{\rm m}$, forming the state or analysis vector $\mathbf{x} = [W; Dm]$. The observations used are $Z_{\rm H}, Z_{\rm DR}$, and $\Phi_{\rm DP}$, yielding the observation vector $\mathbf{y} = [Z_{\rm H}; Z_{\rm DR}; \Phi_{\rm DP}]$. $\rho_{\rm hv}$ is not used because of its limited dynamic range for rain at S-band (Fig. 1f). $\rho_{\rm hv}$ would not add value to retrieval for pure rain as its error is approximately equal to dynamic range. For the solution to (17), let us look at each term individually. First, assume that it is solved azimuthally and that the number of range gates in one azimuth is n_r . The variational retrieval is solved azimuthally (1D in the radial direction) to coincide with Φ_{DP} measurements (i.e., to coincide with Φ_{DP} defined as the integral of K_{DP} over the propagation path). Therefore, the dimension of state variables is $n = 2n_r$ and the dimension of observations is $p = 3n_r$.

251 \mathbf{x}_i is an analysis vector at iteration i with dimension n x 1 (2n_r x 1). It is the concatenate of 252 the analysis vectors of the state variables, *W* and *D*_m, at iteration *i*.

253 \mathbf{x}_{i+1} is an analysis vector at iteration *i*+1 with dimension n x 1 (2n_r x 1). It is the 254 concatenate of the analysis vectors of the state variables, *W* and *D*_m, at iteration *i*+1.

255 \mathbf{x}_b is a background state vector (also known as the first guess) with dimension n x 1 (2n_r x 256 1). It is the concatenate of the background vectors of the state variables, *W* and *D*_m. For this 257 study, the background state vector is obtained from empirical formulas for *W* and *D*_m (Zhang 258 2016):

259
$$W = (1.023 \times 10^{-3}) Z_{\rm h} \times 10^{-0.0742 Z_{\rm DR}^3 + 0.511 Z_{\rm DR}^2 - 1.511 Z_{\rm DR}}$$
(18)

$$D_{\rm m} = 0.0657 Z_{\rm DR}^3 - 0.332 Z_{\rm DR}^2 + 1.090 Z_{\rm DR} + 0.689$$
(19)

261 **y** is an observation vector with dimension p x 1 ($3n_r x 1$). It is the concatenate of the 262 observation vectors for Z_H , Z_{DR} , and Φ_{DP} .

B is a covariance matrix of background errors with dimension n x n ($2n_r \times 2n_r$). The background error standard deviations used in this study were $\sigma_W = 0.707$ g m⁻³ and $\sigma_{Dm} = 1$ mm (i.e., the variances are $\sigma_W^2 = 0.5$ g² m⁻⁶ and $\sigma_{Dm}^2 = 1$ mm²). The relatively large background error gives more weight to observations, which is a reasonable assumption since this is an observation-based retrieval with no model background. The background covariance assumes a Gaussian correlation model (Huang 2000):

269
$$b_{ij} = \sigma_b^2 \exp\left[-\frac{1}{2}\left(\frac{r_{ij}}{r_L}\right)^2\right]$$
(20)

where σ_b^2 is the is the background error covariance (σ_W^2 and σ_{Dm}^2), r_{ij} is the distance between the *i*th and *j*th radar gates, and r_L is the spatial decorrelation length. In this study, r_L was set to 1000 m. An increase (decrease) in the spatial decorrelation length results in an increase (decrease) in smoothing of the final analysis. Therefore, the spatial influence of the observations is determined by the background error covariance matrix.

R is a covariance matrix of observations errors with dimension p x p ($3n_r x 3n_r$). The observation error standard deviations used in this study were $\sigma_{ZH} = 1$ dB, $\sigma_{ZDR} = 0.2$ dB, and $\sigma_{\Phi DP} = 5^{\circ}$. It is assumed that each observation's error is independent from the other observations. Therefore, **R** is a diagonal matrix.

279 *H* is the observation operator that converts state variables in the analysis vector at 280 iteration *i* to observations. These operators (11), (13), and (14) were derived in Section 2.

H is the Jacobian, which contains the partial derivative of each observation operator with respect to the state variables. It has a dimension $p \ge n$ ($3n_r \ge 2n_r$). These can be calculated analytically by taking the partial derivatives of (11), (13), and (14) with respect to *W* and *D*_m:

284
$$\frac{\partial Z_{\rm H}}{\partial W} = \frac{10}{W \ln(10)}$$
(21)

285
$$\frac{\partial Z_{\rm H}}{\partial D_m} = \frac{20(20.87 + 2 * 46.04D_{\rm m} - 3 * 6.403D_{\rm m}^2 + 4 * 0.2248D_{\rm m}^3)}{\ln(10)(-0.3078 + 20.87D_{\rm m} + 46.04D_{\rm m}^2 - 6.403D_{\rm m}^3 + 0.2248D_{\rm m}^4)}$$
(22)

$$\frac{\partial Z_{\rm DR}}{\partial W} = 0 \tag{23}$$

287
$$\frac{\partial Z_{\rm DR}}{\partial D_{\rm m}} = \frac{10(-0.1430 + 2 * 0.3165D_{\rm m} - 3 * 0.06498D_{\rm m}^2 + 4 * 0.004163D_{\rm m}^3)}{\ln(10)(1.019 - 0.1430D_{\rm m} + 0.3165D_{\rm m}^2 - 0.06498D_{\rm m}^3 + 0.004163D_{\rm m}^4)}$$
(24)

288
$$\frac{\partial K_{\rm DP}}{\partial W} = 0.009260 - 0.08699D_{\rm m} + 0.1994D_{\rm m}^2 - 0.02824D_{\rm m}^3 + 0.001772D_{\rm m}^4$$
(25)

13

289
$$\frac{\partial K_{\rm DP}}{\partial D_{\rm m}} = W \left(-0.08699 + 2 * 0.1994 D_{\rm m} - 3 * 0.02824 D_{\rm m}^2 + 4 * 0.001772 D_{\rm m}^3 \right)$$
(26)

The partial derivative of $\partial Z_{DR}/\partial W$ is zero because Z_{DR} only depends on D_m . For the partial derivatives $\partial Z_H/\partial W$, $\partial Z_H/\partial D_m$, and $\partial Z_{DR}/\partial D_m$, the matrix is diagonal because they have gate-togate independence. For $\partial \Phi_{DP}/\partial W$ and $\partial \Phi_{DP}/\partial D_m$, the matrix is a lower triangle because Φ_{DP} is defined as two times the summation of K_{DP} up to a given gate:

294
$$\phi_{\rm DP}(r_n) = 2 \sum_{i=1}^N K_{\rm DP}(r_i) \Delta r_i$$
(27)

The observation-based variational retrieval is complete when the analysis vector, \mathbf{x}_{i} , converges to a solution. A block diagram of the iteration procedure for the variational retrieval is shown in Figure 2. In practice, the number of iterations can be fixed and/or a convergence test applied. If there are nonrain hydrometeors present (e.g., hail) anywhere in the azimuth, there may not be a solution or an unrealistic solution due to contamination within the azimuth (since the retrieval is solved azimuthally). This is because these parametrized polarimetric radar forward observation operators were derived using the T-matrix calculations for rain.

To solve this for an entire plan position indicator (PPI) radar scan, each azimuth within a PPI scan is solved independently. A PPI scan is a radar scan where the elevation angle is constant and the azimuth angle varies. In other words, simply loop through all the azimuths to get a solution for an entire radar elevation scan.

306

307 4. Testing the Variational Retrieval on DSD Data

The variational retrieval and observation operators are tested by conducting experiments on 2DVD data collected on 2005 May 13 (Zhang 2015). The 2DVD data are from a quasi-linear convective system (QLCS) that has an estimated storm motion from 16 to 18 m s⁻¹ or ~1 km min⁻ ¹. The 2DVD data are collected for ~4.63 hours; however, only the results for the first 60 min are shown because this is when the leading convective-line of the QLCS passes over the disdrometer and when the most significant increase in Φ_{DP} occurs (i.e., the most notable part of the dataset).

314 First, the intrinsic or true values for W and $D_{\rm m}$ are calculated using the 2DVD data. When 315 calculating the intrinsic W and $D_{\rm m}$, 5-point averages of the raw W and $D_{\rm m}$ calculations are used to 316 remove noise from the data. Second, $Z_{\rm H}$, $Z_{\rm DR}$, and $\Phi_{\rm DP}$ are calculated using the observation 317 operators (11), (13), and (14) and the relationship between Φ_{DP} and K_{DP} (27). These calculated 318 polarimetric variables are assumed to be the true observations in these experiments. Third, it is 319 assumed that the time-series of 2DVD data represents a radial of radar data. This is assumed 320 because the storm motion remained nearly steady-state. Based on the storm motion, each minute 321 of 2DVD data represents approximately a 1-km range gate of data. Thus, the entire 60 min of 322 data represents a radial that is 60 km in length. Finally, in all of these experiments, a constant 323 background is calculated using the empirical relationship given in (18) and (19). The mean of the 324 empirically-derived W and $D_{\rm m}$ are calculated using the entire radial of data (~4.63 hours). These 325 mean values for W and D_m are used as the constant background.

326

327 a. OI vs. nonlinear

In the first experiment, the OI solution (16) is compared to the nonlinear solution (17). Both of these solutions only include $Z_{\rm H}$ and $Z_{\rm DR}$ ($\Phi_{\rm DP}$ is not included). The relative advantage of the nonlinear solution over OI is demonstrated in Figure 3. For all analyses, the nonlinear solution is closer to the truth. For example, the peak values of *W* are 0.65 g m⁻³ and 1.58 g m⁻³ for the OI and nonlinear solutions at 20 km, respectively (Fig. 3a). The nonlinear solution is closer to the true maximum value of 1.94 g m⁻³.

334 For Z_H, there is generally an underestimation of values in the OI solution when compared 335 to the nonlinear solution and truth (Fig. 3b). For example, the peak values are ~45 dBZ and ~50 336 dBZ for the OI and nonlinear solutions at 29 km, respectively. The nonlinear solution is the same 337 as the truth in this case. $D_{\rm m}$ and $Z_{\rm DR}$ are similar for both the OI and nonlinear solutions with an 338 exception of the peak values (Fig. 3c, d). There a slight underestimation on the OI solution at peak values. For example, the true value at 12 km is 3.05 dB for Z_{DR} and 3.00 mm for D_m. This 339 340 compares to 2.85 dB and 2.80 mm for the OI solution and 3.00 dB and 2.94 mm for the nonlinear 341 solution.

The advantage of the nonlinear solution over OI is also shown by the final value of Φ_{DP} . Since Φ_{DP} is integrated in the along-radial direction, the value at the final gate provides insight on the performance across the entire radial of data. In this experiment, the final value of Φ_{DP} for the OI solution at ~15° is nearly half when compared to the nonlinear solution at ~29° (Fig. 3e). The nonlinear solution is very close to the true value, which is 30°. This seems reasonable and intuitive based on the significant underestimation of *W* across the entire radial with the OI solution.

349

350 b. Nonlinear without Φ_{DP} vs. nonlinear with Φ_{DP}

In the second experiment, the nonlinear solution without Φ_{DP} is compared to that with Φ_{DP} . The advantage of including Φ_{DP} in the variational retrieval is shown in Figure 4. The most evident advantage is seen in *W*, especially with larger values (>1.5 g m⁻³) (Fig. 4a). The peak values of *W* are 1.58 g m⁻³ and 1.75 g m⁻³ for nonlinear solution without and with Φ_{DP} , respectively. The nonlinear solution with Φ_{DP} is closer to the true maximum value of 1.94 g m⁻³. For $Z_{\rm H}$, $D_{\rm m}$, $Z_{\rm DR}$, the benefits of including $\Phi_{\rm DP}$ are negligible (Fig. 4b, c, and d). The final value of $\Phi_{\rm DP}$ for the nonlinear solution without $\Phi_{\rm DP}$ is ~29°, which is close to the nonlinear solution with $\Phi_{\rm DP}$ and the truth at ~30° (Fig. 4e). When compared to the previous experiment, the final value of $\Phi_{\rm DP}$ does not have a significant difference.

Even though most of the analyses show little or no benefit, it is important to note that there is still evidence of some benefit of adding Φ_{DP} in addition to Z_H and Z_{DR} as shown with the increase in the peak *W* value. This improves results for *W* that is closer to the true value. In other words, the inclusion of Φ_{DP} is important for the retrieval of larger *W* values.

364

365 *c. Simulated observations*

In the third experiment, the nonlinear solution with Φ_{DP} is compared to the nonlinear solution with Φ_{DP} with random error added to the observations. Adding random error may result in worse retrievals, which is an important consideration since true radar observations include random error. The random errors included are ± 1 dBZ for Z_H, ± 0.2 dB for Z_{DR}, and $\pm 5^{\circ}$ for Φ_{DP} . These values were chosen because they are used in the covariance matrix of observations errors, as described in previous section.

As shown in Figure 5, this experiment demonstrates that the nonlinear variational retrieval still provides a reasonable solution even with random error in the observations. This is especially true when looking at the solution as a whole. As previously mentioned, the best way to assess this is by looking at the final Φ_{DP} value because Φ_{DP} is an integrated quantity. The final value of Φ_{DP} with random error is the same as truth (~30°) (Fig. 5e). Therefore, on average, the solution is still reasonable.

5. Applying the Variational Retrieval to Radar Data

380 In the previous section, the variational retrieval and observation operators were 381 successfully tested on 2DVD data and simulated observations. The next step is to apply them to 382 real radar data. For this study, the variational retrieval is applied to the PRD collected from the 383 S-band KOUN Weather Surveillance Radar-1988 Doppler (WSR-88D) on 15 June 2011. The radar data are obtained from the National Centers for Environmental Information (NOAA/NCEI 384 385 2011). The radar data from this event captures observations of a downburst and its parent 386 thunderstorm. Only the PRD at a range up to ~ 32 km is used for the variational retrieval because 387 this is a sufficient range to capture the storm of interest. More information about the downburst 388 event can be found in studies by Kuster et al. (2016) and Mahale et al. (2016).

389

a. Single-azimuth experiment

391 The variational retrieval is tested on a single azimuth of radar data before applying it to 392 an entire PPI scan. As previously mentioned, the variational retrieval is solved azimuthally (1D) 393 to coincide with Φ_{DP} observations. As with the 2DVD data, some assumptions and preprocessing to the data are applied to this experiment. A constant background for W and $D_{\rm m}$ are 394 calculated using the azimuthal mean of empirical relationship given in (18) and (19). Also as 395 396 with 2DVD data, quality-control of the radar data is accomplished through a median filter to reduce the noise in the data. In addition, some constraints are placed on the radar data. The Z_H 397 398 minimum is set to 10 dBZ, Z_{DR} is limited to 0.1 to 6 dB, and only positive values of Φ_{DP} are 399 allowed. These constraints are based on the bounds of the observation operators as shown in 400 Figure 1. Finally, a simple attenuation correction is also implemented to the radar data using the

401 differential phase (DP) attenuation correction procedure (Bringi et al. 1990) with coefficients
402 derived in Zhang (2016).

In this experiment, PRD from an azimuth (~3.5°) with moderate $Z_{\rm H}$ (~30 to 50 dBZ) and high $\rho_{\rm hv}$ (>0.98) values is chosen. Based on these observations, the assumption is that only raindrops associated moderate rainfall are present in this azimuth with no hail contamination. This makes it an ideal azimuth to test the variational retrieval. Once *W* and $D_{\rm m}$ are solved using the variational retrieval, the observation operators are used to calculate $Z_{\rm H}$, $Z_{\rm DR}$ and $\Phi_{\rm DP}$ analyses as in the 2DVD experiments.

The W, $D_{\rm m}$, $Z_{\rm H}$, $Z_{\rm DR}$, and $\Phi_{\rm DP}$ analyses for the single azimuth are shown in Figure 6. For comparison, the empirical relationships (18) and (19) are used to calculate W and $D_{\rm m}$. The empirical relationships only depend on $Z_{\rm H}$ and $Z_{\rm DR}$ and do not account for $\Phi_{\rm DP}$. Even so, the empirical relationships can be used to qualitatively assess the results of the variational retrieval. The observed values for $Z_{\rm H}$, $Z_{\rm DR}$, and $\Phi_{\rm DP}$ are shown as a comparison as well.

For *W*, the variational retrieval is smoother than the empirical relationship (Fig. 6a). This is reasonable because the variational retrieval takes account for the spatial covariance, which is a function of the spatial decorrelation length. The variational retrieval would be less (more) smooth with a smaller (larger) spatial decorrelation length. The local maxima and minima for *W* are at approximately the same range; however, the magnitudes of these maxima and minima are sometimes different. For example, at ~9 to 10 km range, the maximum magnitude of the difference between the two *W* values is 0.43 g m⁻³.

421 This range is also where there is a noticeable difference between the $Z_{\rm H}$ analysis and 422 observations (Fig. 6b). At the same range, the maximum magnitude of the difference between $Z_{\rm H}$ 423 analysis and observations is 3.8 dB. The $\Phi_{\rm DP}$ observations are essentially increasing W values

beyond simply what the $Z_{\rm H}$ values would suggest in this range, which results in larger $Z_{\rm H}$ values in the analysis to keep it consistent with the $Z_{\rm H}$ observation operator. If the $\sigma_{\Phi \rm DP}$ was increased (decreased), the *W* values in this range would decrease (increase) because of decreasing (increasing) influence of $\Phi_{\rm DP}$ with more (less) weight given to $Z_{\rm H}$. Outside of this range, the $Z_{\rm H}$ analysis is fairly close to the $Z_{\rm H}$ observations (albeit smoother).

For the $D_{\rm m}$ analysis (Fig. 6c), the variational retrieval is smoother than the empirical relationship as well and the peak values are less than the empirical relationship at a range of ~6 km and ~11 km. This occurs because the observation operators, and more specifically the T-Matrix calculations, have a smaller dynamic range for $D_{\rm m}$. Consequently, the reduction of $D_{\rm m}$ also results in a coincident reduction in $Z_{\rm DR}$ as well (Fig. 6d). Outside these areas, the $Z_{\rm DR}$ analysis is fairly close to the $Z_{\rm DR}$ observations.

When comparing *W* with Φ_{DP} at this range (Fig. 6a, e), it is evident that the Φ_{DP} has a positive impact in modulating *W*. The Φ_{DP} analysis has a relatively steep slope in this range (i.e., relatively large K_{DP}), which is why *W* is greater than in the empirical relationship that does not have the contribution from Φ_{DP} .

It is also noteworthy that the Φ_{DP} analysis can only monotonically increase when using the variational retrieval (Fig. 6e). This is because only positive K_{DP} values are allowed in the observation operator. As shown, Φ_{DP} observations are noisy and chaotic when compared to Z_H and Z_{DR} observations, which is why $\sigma_{\Phi DP}$ is relatively large when compared to σ_{ZH} and σ_{ZDR} . The variational retrieval provides a method to calculate a smoother, monotonically increasing Φ_{DP} while accounting for the Z_H and Z_{DR} observations. This is the characteristic that would be expected for Φ_{DP} in a pure-rain azimuth with no hail contamination at S-band. In addition, because the Φ_{DP} analysis can only monotonically increase with range, Φ_{DP} observations serve as a constraint for the variational retrieval. This is because the final Φ_{DP} analysis value is dependent on the total sum of *W* across the entire azimuth. The other two observations, Z_{H} and Z_{DR} , are independent from gate-to-gate.

450 Overall, the single-azimuth experiment on an area of moderate rainfall was successful. 451 The locations of local maxima and minima for W, D_m , Z_H , and Z_{DR} are similar to either the 452 empirical formula or the observations with some differences in magnitude. The similarity to the 453 empirical formula provides a proof of concept that the variational retrieval yields a reasonable 454 solution. The subtle differences can be accounted for mostly with the addition of $\Phi_{\rm DP}$ 455 observations (in the case of W and $Z_{\rm H}$) and the limitation of T-Matrix calculations used in the 456 observation operators (in the case of $D_{\rm m}$ and $Z_{\rm DR}$). Finally, these results demonstrate that the 457 variational retrieval has utility in calculating a smoother, monotonically increasing (more 458 physically-representing) Φ_{DP} .

459

460 *b. PPI experiment*

461 After the single-azimuth experiment, the variational retrieval was applied on the entire 462 PPI of radar data by looping through all 360 azimuths. As with the single-azimuth experiment, once $D_{\rm m}$ and W are obtained using the variational retrieval, the observation operators are used to 463 calculate Z_H and Z_{DR} analyses to assess the observation operators in context to the forward 464 465 problem. Instead of calculating Φ_{DP} , the K_{DP} analysis is shown to demonstrate it can be estimated 466 using this method. K_{DP} is operationally useful because it is better correlated to rainfall rates due 467 to it being unaffected by calibration error and attenuation as well as being more linearly related 468 to rainfall rate, which gives it utility for quantitative precipitation estimation (QPE) (Bringi and

469 Chandrasekar 2001). The *W* and $D_{\rm m}$ analyses are shown in Figure 7 and the $Z_{\rm H}$ and $Z_{\rm DR}$ 470 observations and analyses are shown in Figure 8. The differences between the observations and 471 analyses are also shown (Fig. 8e, f). The $K_{\rm DP}$ analysis is shown in Figure 9.

472 The most evident feature in the analyses is the noisy results across the western to northwestern sector of the PPI. Parts of the analyses in this area are unrealistic and the Z_H 473 analysis differs significantly from the observations. There are even some azimuths that are 474 completely void of data in the analyses. Further investigation was done on why the variational 475 476 retrieval failed in this sector and will be discussed later in this section. In addition to this sector, 477 there are a few other azimuths that are void of data. These azimuths do not have a solution likely due to low Z_H values and/or the presence of nonmeteorological scatterers (i.e., biological or 478 479 ground clutter) in the azimuth.

480 Outside of these areas, the $Z_{\rm H}$ and $Z_{\rm DR}$ analyses are similar to the observations (Fig. 8). The local maxima and minima are near the same locations (with an occasional variation in the 481 482 magnitude), which results in same precipitation structure. Some variability in magnitude is 483 expected because an optimal analysis that utilizes all the information from $Z_{\rm H}$, $Z_{\rm DR}$, and $\Phi_{\rm DP}$. 484 Also, some smoothing occurs because the spatial covariance is taken into account. One 485 noticeable difference is that the peak magnitude of Z_{DR} is tempered when compared to 486 observations. As noted in the single-azimuth experiment, this is because of the dynamic range of 487 $D_{\rm m}$ that can be calculated through the variational retrieval.

There are also slight discontinuities from azimuth to azimuth. This is likely an artifact from changes in the errors of Φ_{DP} observations from azimuth to azimuth and the subsequent effect on the final Φ_{DP} analysis value. In this study, the Φ_{DP} error distribution is assumed constant across the entire PPI, which may not be the case. As previously noted, the final Φ_{DP} 492 analysis value is dependent on the total sum of *W* across the entire azimuth. Slight deviations in 493 the final Φ_{DP} analysis value will result in a slight change for the retrieved state variables across 494 the entire azimuth.

The $K_{\rm DP}$ analysis is an optimal analysis that utilizes information from $Z_{\rm H}$ and $Z_{\rm DR}$ (in addition to $\Phi_{\rm DP}$) (Fig. 9). As previously noted, $\Phi_{\rm DP}$ can only monotonically increase when using the variational retrieval. This is a constraint that comes out of the observation operator. Therefore, negative $K_{\rm DP}$ values cannot exist. This is reasonable for areas of pure rain. Like $Z_{\rm H}$ and $Z_{\rm DR}$, $K_{\rm DP}$ is relatively smooth from the variational retrieval.

500 Applying a simplified hydrometeor classification algorithm (HCA) from Mahale et al. 501 2014 (based upon Park et al. 2009) provided more insight on why the variational retrieval failed 502 on the western to northwestern sector of the PPI. The HCA discriminates between 10 classes of 503 radar echo: 1) ground clutter and anomalous propagation (GC/AP), 2) biological scatterers (BS), 504 3) dry aggregated snow (DS), 4) wet snow (WS), 5) crystals (CR), 6) graupel (GR), 7) big drops (BD), 8) light and moderate rain (RA), 9) heavy rain (HR), and 10) a mixture of rain and hail 505 506 (RH). As shown in Figure 10, these radials had radar gates that are classified as RH, which 507 implies there was potentially hail contamination on these radials. This classification seems 508 reasonable based on the observed Z_H and Z_{DR} values. Relatively-high Z_H with near-zero or 509 negative Z_{DR} has been observed as a signature associated with hail for S-band radars. In addition, 510 as previously mentioned, the HCA also indicates GC/AP and/or BS in some of the other 511 azimuths where no solution was found.

As noted, if there is hail contamination (or any nonrain hydrometers) present anywhere in the azimuth, there may not be a solution or an unrealistic solution due to contamination. This is because the observation operators were derived using the T-matrix calculations for pure rain. On

Figures 7-9, the sector associated with hail contamination is highlighted using the results from the HCA. For operational purposes in the detection of hazardous weather, the failure of the variational retrieval could actually be a tool used to determine if hail is present in a thunderstorm. This would be especially true if the variational retrieval is conducted locally over a set number of simultaneous gates rather than an entire azimuth.

520

521 6. Summary and Conclusions

In this study, a proposed method for the observation-based variational retrieval of the physical parameters of W and D_m from S-band PRD was developed through the use of parametrized polarimetric radar observation operators and a nonlinear, iterative method.

The observation operators were derived for $Z_{\rm H}$, $Z_{\rm DR}$, and $K_{\rm DP}$ using the T-matrix method and an exponential distribution model, which link PRD to $D_{\rm m}$ and W. These simple observation operators are advantageous because they include relevant physics, are numerically efficient, and have easily calculated first derivatives. In addition, these parameters can be easily assimilated to NWP because are directly related to the prognostic physics variables of $N_{\rm t}$ and $q_{\rm r}$.

530 The use of an exponential distribution results in an overdetermined problem, which is 531 advantageous in this retrieval because $\Phi_{\rm DP}$ is used as a constraint along the azimuth (i.e., it can 532 only monotonically increase), the PRD are not all independent (e.g., Z_{DR} has dependence on Z_{H}), 533 and there are measurement and model errors. For example, this would make the retrieval less 534 sensitive to Z_{DR} bias because Z_H , Z_{DR} , and K_{DP} are all a function of D_m (though some error will be 535 added to the retrieval due to the assumption that observations are unbiased). The disadvantage of 536 an exponential distribution is that it may not perform as well in areas where there is concavity 537 present in the DSD. Nevertheless, recent work has shown the two parameter DSD model has an

advantage for retrieving DSDs from PRD when compared to the three-parameter Gamma DSDmodel (Huang et al. 2019).

540 Using these observation operators, the Gauss-Newton iterative method for moderately 541 nonlinear problems was successively used in the variational retrieval. Experiments on calculated 542 W and $D_{\rm m}$ from 2DVD data demonstrated the advantages of using the nonlinear solution when 543 compared to the OI solution and including Φ_{DP} observations in addition to Z_{H} and Z_{DR} . Φ_{DP} may 544 have only provided a slight benefit because it does not increase substantially at S-band 545 (especially for light to moderate rainfall). It was also shown even with the inclusion of random 546 error into the 2DVD data, a reasonable solution can still be found. These advantages were 547 discussed in context of the both the retrieval of W and $D_{\rm m}$ and the estimation of polarimetric 548 radar variables to test the observation operators both inverse and forward, respectively. It can be 549 concluded from these experiments that accounting for nonlineary and including $\Phi_{\rm DP}$ observations 550 is beneficial for the retrieval W and $D_{\rm m}$. The nonlinearity is noteworthy because even though 551 most problems are not truly linear, the data assimilation community often assumes linearity.

When applied to real radar data, the variational method yields promising results for the retrieved *W* and $D_{\rm m}$ and derived $K_{\rm DP}$ within areas of pure rain. Since the observation operators were derived with the assumption of pure rain, the retrieval failed where there was hail contamination. In this case, the derived $K_{\rm DP}$ from *W* and $D_{\rm m}$ can be considered a related parameter that describes the rain microphysics.

557 Overall, the successful variational retrieval of W and D_m from the 2DVD and the real 558 radar data from two separate events demonstrate the utility of the proposed method. This method 559 develops a framework that utilizes proven techniques from the data assimilation community. 560 While the inclusion of a background state may not be a significant advantage for 1D (azimuth) retrieval, it can be important for 2D or 3D retrieval problems, which can become underdetermined if the background is not included because of the data distribution/coverage and spacing.

564 In the future, additional studies can be conducted on simulated and real data using this 565 proposed retrieval method. This may include controlled experiments where we know the true 566 quality of the measurements and their error. Other future research includes deriving the 567 observation operators for other radar frequencies (e.g., X- and C-band) and incorporating 568 observation operators for other hydrometer types such as hail. The inclusion of other 569 hydrometeors is important for operational use, especially when retrieving information above the 570 melting layer. For X- and C-band, attenuation would have to definitely be accounted for in the 571 Z_H and Z_{DR} observation operators (or applied prior to the retrieval). Finally, a goal is to utilize the 572 variational retrieval for data assimilation in NWP to improve model forecasts.

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Fig. 1. The fitted parameterized polarimetric forward observation operators compared to direct calculations from T-matrix method for (a) Z_h , (b) Z_H , (c) Z_{dr} , (d) Z_{DR} , (e) K_{DP} , and (f) ρ_{hv} . These are normalized for *W* of 1 g m⁻³. The equation on each panel is the derived observation operator.

Fig. 2. Block diagram of the iteration procedure for the nonlinear variational retrieval for a singleazimuth of radar data.

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Fig. 3. OI analysis and nonlinear analysis compared to the truth for (a) W, (b) $Z_{\rm H}$, (c) $D_{\rm m}$, (d) $Z_{\rm DR}$, and (e) $\Phi_{\rm DP}$ using 2DVD data collected on 2005 May 13. The constant background field for W

678 and $D_{\rm m}$ are also shown.

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Fig. 4. Same panels as Figure 3, but with nonlinear analysis without Φ_{DP} and nonlinear analysis with Φ_{DP} compared to the truth.

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Fig. 5. Same panels as Figure 3, but with nonlinear analysis without error and nonlinear analysis with random error compared to the truth. The simulated observation (truth + random error) for $Z_{\rm H}$, $Z_{\rm DR}$, and $\Phi_{\rm DP}$ are also shown.

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Fig. 6. Analysis for (a) W, (b) $Z_{\rm H}$, (c) $D_{\rm m}$, (d) $Z_{\rm DR}$, and (e) $\Phi_{\rm DP}$ by applying the variational retrieval on a single azimuth of 5.3° elevation scan of KOUN WSR-88D data from 15 June 2011 at 0020 UTC. The observed values for $Z_{\rm H}$, $Z_{\rm DR}$, and $\Phi_{\rm DP}$ are plotted for comparison. The empirical relationship and constant background field for W and $D_{\rm m}$ are shown as well. Fig. 7. Analysis for (a) W and (b) D_m by applying the variational retrieval using a 5.3° elevation scan of KOUN WSR-88D data from 15 June 2011 at 0020 UTC. The boundaries of the sector with hail contamination are marked by the black lines.

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Fig. 8. (a) $Z_{\rm H}$ observations, (b) $Z_{\rm DR}$ observations, (c) $Z_{\rm H}$ analysis, (d) $Z_{\rm DR}$ analysis, (e) $Z_{\rm H}$ observations and analysis difference, and (f) $Z_{\rm DR}$ observations and analysis difference using a 5.3° elevation scan of KOUN WSR-88D data from 15 June 2011 at 0020 UTC. The analyses are from applying the variational retrieval.

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Fig. 9. K_{DP} analysis from the variational retrieval using a 5.3° elevation scan of KOUN WSR-88D data from 15 June 2011 at 0020 UTC. The boundaries of the sector with hail contamination are marked by d black lines.

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Fig. 10. HCA applied to 5.3° elevation scan of KOUN WSR-88D data from 15 June 2011 at
0020 UTC. The classifications used in HCA are defined in section 5b. The boundaries of the
sector with hail contamination are marked by the black lines.



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Fig. 1. The fitted parameterized polarimetric forward observation operators compared to direct calculations from T-matrix method for (a) Z_h , (b) Z_H , (c) Z_{dr} , (d) Z_{DR} , (e) K_{DP} , and (f) ρ_{hv} . These are normalized for *W* of 1 g m⁻³. The equation on each panel is the derived observation operator.



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Fig. 2. Block diagram of the iteration procedure for the nonlinear variational retrieval for a single

716 azimuth of radar data.



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Fig. 3. OI analysis and nonlinear analysis compared to the truth for (a) W, (b) $Z_{\rm H}$, (c) $D_{\rm m}$, (d) $Z_{\rm DR}$, and (e) $\Phi_{\rm DP}$ using 2DVD data collected on 2005 May 13. The constant background field for Wand $D_{\rm m}$ are also shown.



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Fig. 4. Same panels as Figure 3, but with nonlinear analysis without Φ_{DP} and nonlinear analysis with Φ_{DP} compared to the truth.



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Fig. 5. Same panels as Figure 3, but with nonlinear analysis without error and nonlinear analysis with random error compared to the truth. The simulated observation (truth + random error) for $Z_{\rm H}$, $Z_{\rm DR}$, and $\Phi_{\rm DP}$ are also shown.



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Fig. 6. Analysis for (a) W, (b) $Z_{\rm H}$, (c) $D_{\rm m}$, (d) $Z_{\rm DR}$, and (e) $\Phi_{\rm DP}$ by applying the variational retrieval on a single azimuth of 5.3° elevation scan of KOUN WSR-88D data from 15 June 2011

- at 0020 UTC. The observed values for $Z_{\rm H}$, $Z_{\rm DR}$, and $\Phi_{\rm DP}$ are plotted for comparison. The
- empirical relationship and constant background field for W and D_m are shown as well.



Fig. 7. Analysis for (a) W and (b) $D_{\rm m}$ by applying the variational retrieval using a 5.3° elevation scan of KOUN WSR-88D data from 15 June 2011 at 0020 UTC. The boundaries of the sector with hail contamination are marked by the black lines.



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Fig. 8. (a) $Z_{\rm H}$ observations, (b) $Z_{\rm DR}$ observations, (c) $Z_{\rm H}$ analysis, (d) $Z_{\rm DR}$ analysis, (e) $Z_{\rm H}$ observations and analysis difference, and (f) $Z_{\rm DR}$ observations and analysis difference using a

- 5.3° elevation scan of KOUN WSR-88D data from 15 June 2011 at 0020 UTC. The analyses are
- 743 from applying the variational retrieval.





Fig. 9. K_{DP} analysis from the variational retrieval using a 5.3° elevation scan of KOUN WSR-88D data from 15 June 2011 at 0020 UTC. The boundaries of the sector with hail contamination are marked by d black lines.



750

Fig. 10. HCA applied to 5.3° elevation scan of KOUN WSR-88D data from 15 June 2011 at
0020 UTC. The classifications used in HCA are defined in section 5b. The boundaries of the
sector with hail contamination are marked by the black lines.