Physical Mechanics METR2103
November 2, 2000

Study Guide for Exam #2

The information even below is meant to serve as a guide to help you to prepare for the second hour exam. The absence of a topic or point of discussion DOES NOT MEAN that those topics will not be examined. You should carefully study all lecture notes and distributed reading materials, and re-work quizzes and problem sets.

Conservation and Advection Equations

- Understand local/Eulerian and total/Lagrangian time derivatives and their their physical meaning
- Know the definiton of spatial advection and their physical
- Be able to apply these concepts to problem solving
- Understand the rate of change in a moving framwork and its relation to location time derivative and spatial gradient
- Know the essential componented needed for advection

Vector Calculus and Associated Physical Concepts

- Definition of a vector, magnitude, cross product, dot product, curl, gradient, divergence. Understand what is meant by the projection of a vector.
- Be able to compute the total derivative of a vector
- Given the position vector, know how to obtain the velocity and acceleration Understand uniform circular motion
- Be able to define absolute, relative, and coordinate system motion in relation to one another and solve problems using them.
- Be able to solve line integrals (e.g., work, circulation)
- Know the relationship between a vector and the gradient of its potential function; be able to find the potential, given the force, and vice versa.
- Be able to compute the vertical velocity using the so-called kinematic method
- Be able to apply Gauss' divergence theorem and Stokes' theorem. -- also, understand their physical meaning.
- Understand solid body rotation and how to compute the circulation for an object undergoing it.
- Be able to compute the vorticity and divergence given a set of wind observations.

Motion in the 2-D Plane

- Understand conservative forces and their unique properties insofar as work and potential are concerned.
- Be able to show whether a force is conservative.
- Be able to compute the potential function given its force (see above) and vice versa.
- Be able to solve problems with projectiles

Summary for Lecture Notes

Advection

Lagrangian Framework

Eulerian Framework

For a scalar field *F[x(t), y(t), z(t), t]*

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} + \frac{\partial F}{\partial t}$$

$$= \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z}$$

→ $$\frac{\partial F}{\partial t} = \frac{dF}{dt} - \left(u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} \right)$$

Definition and physical meaning of each terms

Local and total rates of change
Spatial advection – mathematical formula and physical interpretation

Vector form of the advection equation $\frac{\partial F}{\partial t} = \frac{dF}{dt} - \vec{V} \cdot \nabla F$

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k} = \text{advectionvelocity}$$

$$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} = \text{gradientopeartor- a vector}$$

Use of finite difference to estimate spatial derivatives / gradient. E.g., using centered difference

$$\frac{\partial T}{\partial x} \simeq \frac{T(x+\Delta x) - T(x-\Delta x)}{2\Delta x}$$

or more generally

$$\frac{\partial T}{\partial x} \simeq \frac{T(x_1) - T(x_2)}{x_2 - x_1}$$

Definition and physical interpretation of the spatial gradient of a scalar field

Time rate of change in a moving coordinate, e.g., following a moving vehicle

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{C} \cdot \nabla T$$

Application of the above equations to practical problems.

Vector Calculus

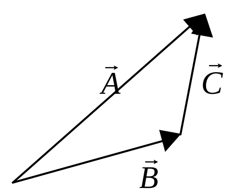
Vector versus scalar

Basic vector operations

Addition and subtraction.

$$\vec{A} = \vec{B} \pm \vec{C} \;\rightarrow\; A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = (B_x \pm C_x)\hat{i} + (B_y \pm C_y)\hat{j} + (B_z \pm C_z)\hat{k}$$

$\vec{A} = \vec{B} + \vec{C} \;\rightarrow$



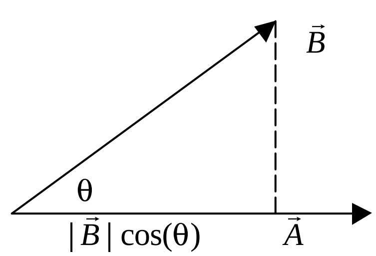
Vector Components in a Given Coordinate:

Unit or Base Vectors and Magnitude/Length of Vector:

Scalar, Dot or Inner Product

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos(q) \qquad (5.5)$$

Physically, one can view the dot product as the projection of one vector onto another.



When dot product is zero, the 2 vectors are $\perp$

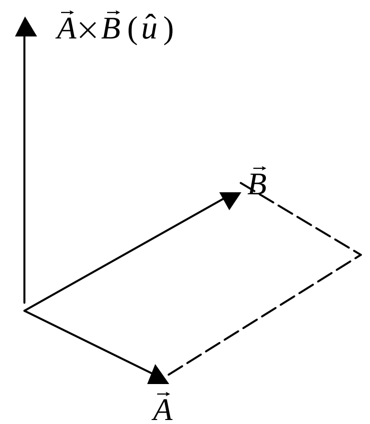
In terms of <u>components</u>,

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z .$$

Vector or "cross" or outer product

$$\vec{A}\times\vec{B} = AB\sin(q)\,\hat{u} = \vec{C} \;=\; \underline{\text{a vector}} \qquad (5.7)$$

Geometric definition:



Right hand rule of cross product.

If $\vec{A} = \vec{B}$ or $\vec{A} \parallel \vec{B}$, then $\vec{A}\times\vec{B}$ =0 - a useful way to see if 2 vectors are parallel.

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= \hat{i}\,(A_y B_z - A_z B_y) - \hat{j}(A_x B_z - A_z B_x) + \hat{k}(A_x B_y - A_y B_x) .$$

An example of the dot product - the definition of work:

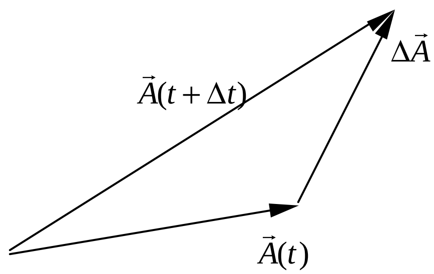
$$W \equiv \int \vec{F}\cdot\vec{dl} = \int F_x dx + F_y dy + F_z dz$$

An example of the cross product – the torque $\vec{t}$ of a force acting on a rotating object:

$$\vec{t} \equiv \vec{r}\times\vec{F}$$

Differentiation and Integration of Vectors

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \vec{A}(t+\Delta t)}{\Delta t}$$



Both change in the magnitude and direction can contribute to the change in a vector.

Position vector and velocity and acceleration vectors

$$\vec{V} = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

$$\vec{V} = \frac{d\vec{r}}{dt} = \hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt} = u\hat{i} + v\hat{j} + w\hat{k}$$

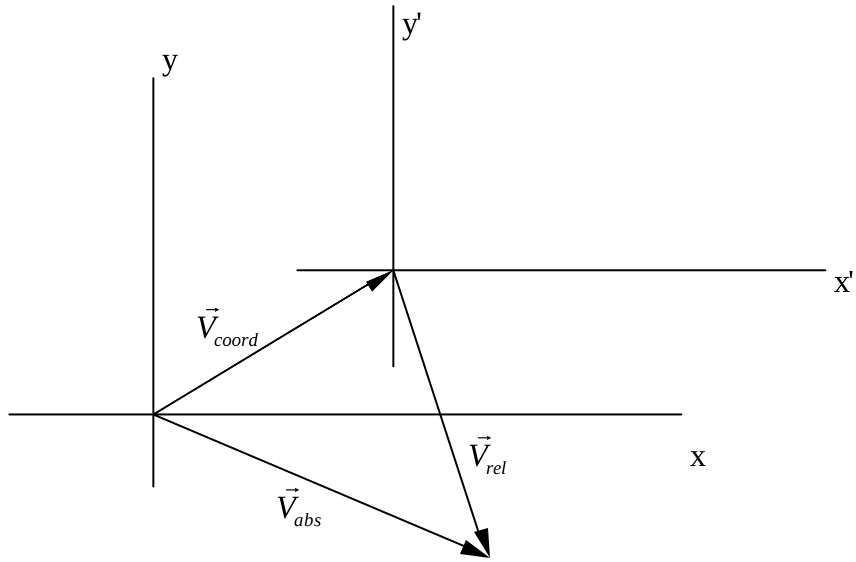
Note that $\vec{V}$ is everywhere tangent to the trajectory of the particle:

The acceleration vector is given by

$$\frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2} = \hat{i}\frac{du}{dt} + \hat{j}\frac{dv}{dt} + \hat{k}\frac{dw}{dt}.$$

Absolute and Relative Motion

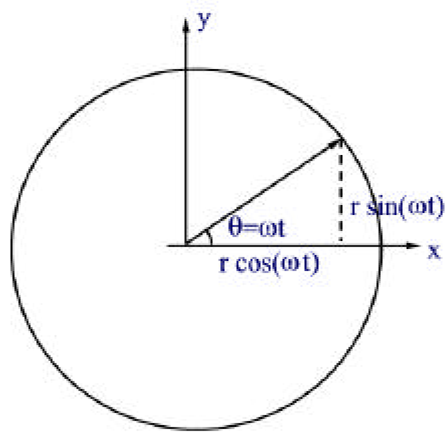
$$\vec{V}_{abs} = \vec{V}_{rel} + \vec{V}_{coord}$$



Let (x,y) be the "fixed" or absolute coordinate system
Let (x', y') be the moving coordinate system

Uniform Circular Motion

With UCM, the length of the position vector is constant and points from the center of circle to the position of the moving object. The direction of position vector changes at a constant angular rotation rate.

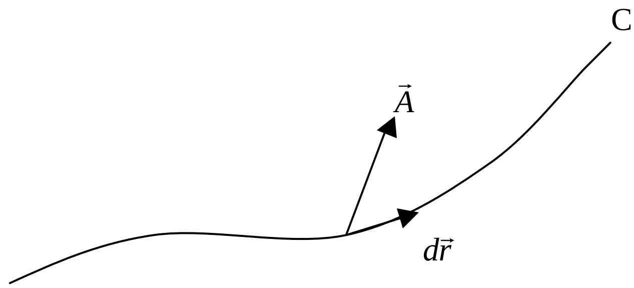


With UCM, the velocity vector is always perpendicular the position vector, and the acceleration always points to the center of circle. It is called centripetal acceleration. This acceleration is real and is due to the constant change in the direction of the velocity vector.

Integration of Vectors

A line integral involves integration of a function (can be a vector field) along a given line path.

Work done by a vector force along a path and circulation along a closed path are two examples of vector line integral.



$d\vec{r}$ is tangential to C

$\int \vec{A}\cdot d\vec{r}$ = projection of $\vec{A}$ onto the various segment $d\vec{r}$ at points along the curve.

$$\int_{P_1}^{P_2} \vec{A}\cdot d\vec{r} = \int_{P_1}^{P_2} A_x dx + A_y dy + A_z dz \,.$$

If s = distance measured along the curve, then

$$\int \vec{A}\cdot d\vec{r} = \int A\cos q\, ds$$

If the curve C is closed, then we write $\oint \vec{A}\cdot d\vec{r}$.

When $\vec{A} = \nabla f$, $\oint \vec{A}\cdot d\vec{r}$ = 0 is always true (remember your quiz problem?), and $\int_{P_1}^{P_2} \nabla f \cdot d\vec{r}$ is independent of the path from P_1 to P_2.

Circulation, defined in the following is another example of vector line integral.

$$C = \oint \vec{V}\cdot d\vec{r} \,.$$

Gradient

Gradient is the result of applying the del operation $del() = \nabla() \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ to a scalar field. Physically, it measures the spatial variation in this scalar field, and points in the direction that this scalar quantity increases in the fastest.

The gradient of a scalar A is a vector, and is defined as

$$grad(A) = \nabla A \equiv \hat{i}\frac{\partial A}{\partial x} + \hat{j}\frac{\partial A}{\partial y} + \hat{k}\frac{\partial A}{\partial z}$$

Divergence

The divergence of a vector field is defined as

$$\mathrm{Div}(\vec{A}) = \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

It is a scalar. It measures the 'rate of expansion' of the vector field. When divergence is negative, it represents contraction, or convergence.

When this vector is the velocity, we are talking about velocity or flow divergence.

For an incompressible flow (the atmosphere is close to being incompressible), we have $\nabla \cdot \vec{V} = 0$, i.e., $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$. It's also called non-divergent. Using this property, and given observations of horizontal winds from which horizontal divergence can be calculated, we can obtained typically unobserved vertical wind by vertically integration the equation. The method is called the 'kinematic" method for obtaining w.

Upward motion occurs in regions with lower-level convergence (e.g., at the surface cold front), while down ward motion is found where low-level divergence is present (e.g., below thunderstorm downdraft).

The Gauss Divergence Theorem relates the divergence in a volume to the net fluxes through the surface that encompasses this volume. The total divergence in the volume is equal to the net fluxes through the surface. Mathematically, it is

$$\iiint_{\Omega} \nabla \cdot \vec{V} d\Omega \equiv \iint_{S} \vec{V} \cdot \vec{n} ds$$

Curl and Vorticity

The curl of vector $\vec{A}$ is defined as

$$\nabla\times\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix} = \hat{i}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)-\hat{j}\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)+\hat{k}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

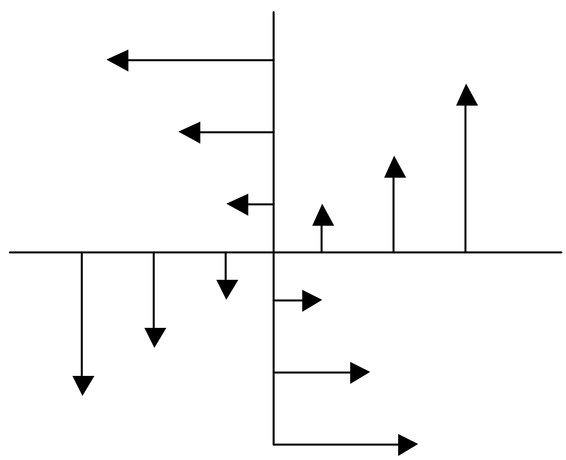
and it is a vector.

When this vector is the velocity, the curl is called vorticity (vector).

The vertical component of vorticity (often called vertical vorticity) is most important in meteorology, just like the horizontal divergence, and it is

$$\zeta=\hat{k}\cdot\left(\nabla\times\vec{V}\right)=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

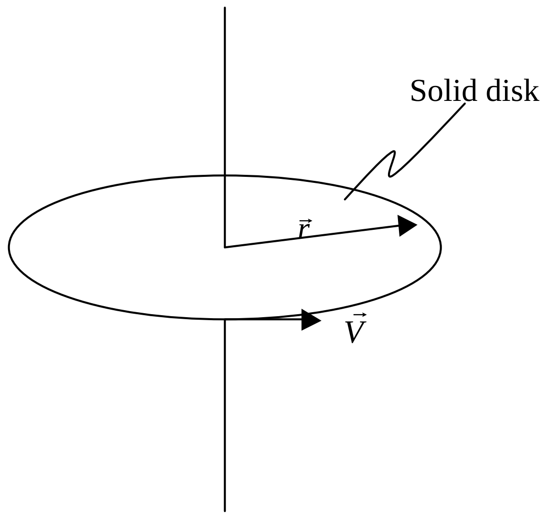
For an object undergoing **solid body rotation**, the velocity components have the form of $u = -\omega y$, $v = \omega x$, and the flow has the following pattern:



The associated vorticity can be found to be $\zeta = \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} = 2\omega$, which is actually 2 times of the rotational rate, ω.

Although the ∇ operator behaves like a vector when is applied to a field, in the curl operation, right hand rule does not apply. The direction of the curl depends on the vector being operated on.

For a solid body rotating at an angular velocity of $\vec{\Omega}$, the velocity given at radus *r* from the rotation axis is given by



$$\vec{V} = \vec{\Omega} \times \vec{r} = \text{tangential velocity}$$

The vorticity can be found as the **circulation** / area enclosed by the circle

$$C = \int \vec{V} \cdot \overrightarrow{dl} = \int (\vec{\Omega} \times \vec{r}) \cdot \overrightarrow{dl}$$

$$= \int_0^{2p} |\vec{\Omega}||\vec{r}|\sin(90^\circ)\, r\ dq = 2p\ \Omega r^2$$

therefore C/A = 2Ω → vorticity for an object in solid body rotation is twice its angular rotation rate, agreeing with our finding earlier based on the velocity function.

Stokes Theorem

Stokes Theorem relatives the total vorticity normal to a surface to the circulation along the path that encompasses this surface:

$$\iint (\nabla \times \vec{V}) \cdot \vec{n}\, ds = \oint \vec{V} \cdot d\vec{r}\ .$$

It says for a given surface enclosed by a curve C, the surface integral of the vorticity, $\nabla \times \vec{V}$, in the direction of the outward normal vector, $\vec{n}$, equals the line integral of velocity along curve C.

Conservative Force

Be definition, a force is said to be conservative if the work done by it is independent of the path, or equivalently, the work done by this force around a closed path is identically zero, i.e., $\oint \vec{F} \cdot \vec{dl} = 0$.

$\nabla \times \vec{F} = 0$, the above condition is satisfied according the Stokes' Theorem.

$$\iint (\nabla \times \vec{V}) \cdot \vec{n}\, ds = \int \vec{V} \cdot d\vec{r} \qquad (6.1)$$

If $\vec{F} = \nabla f$, then $\nabla \times \vec{F} = 0$ is satisfied automatically.

The test that $\nabla \times \vec{F} = 0$ is an easy way to show that a force is conservative.

Gravity is a conservative force.

Solving for Potential Function

Given a conservative force, one can obtained a potential energy from

$$\vec{F} = -\nabla V$$

i.e,

$$F_x = -\frac{\partial V}{\partial x}, \; F_y = -\frac{\partial V}{\partial y} \text{ and } F_z = -\frac{\partial V}{\partial z}$$

by integrating these three first-order differential equations.

Projectile Motion

The vector equation of motion governing a projectile is

$$\frac{d(\vec{V}m)}{dt} = \sum \vec{F}$$

For a single projectile subjecting to gravity only, the equation is

$$m\frac{d^2\vec{r}}{dt^2} = -mg\hat{k}$$

or

$$m\frac{d^2x}{dt^2}=0,$$

$$m\frac{d^2y}{dt^2}=0,$$

$$m\frac{d^2z}{dt^2}=-mg$$

without friction, these three equations are decoupled and can be solved independently. From the solutions, we can derive the trajectory of the projectile and quantities such as maximum height and range that can be reached and the time it takes to reach them.

If you have any problem understanding the above, discuss with your fellow students and come to Sunday evening's help session. But try not to wait until that last minute!!