Example of simple harmonic motion
Physical Mechanics, Fall 2000

This example demonstrates the use of momentum theorem and work-energy theorem, the computation of kinetic and potential energy (from a conservative force), the use of total energy conservation and the equation of motion (based on Newton's second law), the determination of the intrinsic frequency (related to eigenvalue of the problem) of an oscillator, and the determination of arbitrary constants found in the general oscillator solution (from initial conditions). It also shows you how to demonstrate that the total energy is indeed conserved.

You are strongly suggested to do the problem yourself first before looking at the answers!

A 2 kg ball is attached to the end of a sufficiently long spring with a spring constant k = 8 kg s^{-2}. The other end of the spring is attached to a fixed wall. The ball is hit by a hammer in the direction away from the wall. The average force is 40 kg m s^{-2} and the force acted on the ball for 0.1 s.

a) Determine the initial speed of the ball right after it is hit.

b) Determine the maximum length the spring will be stretched.

c) Find the position of the ball at time t by solving the equation of motion

d) Find the velocity of the ball at time t from the position function obtained in c)

e) Determine the position and velocity of the ball at 200 s from the above solution

f) Determine the kinetic energy of the ball at 200 s from the velocity.

g) Determine the potential energy of the ball at 200 s from its definition.

h) Determine the total energy at 200 s and compared it with that at t=0. What does it tell you?

Given

mass m = 2 kg
spring constant k = 8 kg s^{-2}
Impulsive force averaged over a short period of time = 40 kg m s^{-2}
The time period that force acted on the ball Δt = 0.1s

To be found: A whole bunch of things

Answers:

a) Use momentum theorem to determine the initial velocity v_0:

$mv_0 - m\,0 = F\,\Delta t \rightarrow v_0 = F\Delta t/m = 40*0.1/2 = 2$ m/s

b) The ball will be oscillating around the initial position x=0. The maximum length spring is stretched = the amplitude of oscillation = the distance of the ball from the initial position when the velocity of the ball is reduced to zero by the restoring force. At that point, the kinetic energy is zero and potential energy is at a maximum. The total energy at that point should be equal to the total energy at the initial time (or at any time but we know that total energy at the initial time). So we want to use total energy conservation. But before that, we need to find out the potential energy (unless you remember for a spring $V(x) = kx^2/2$).

The restoring force from a spring is F = - kx.

$$V(x) = \int_x^0 F dx = \int_x^0 (-kx)dx = \frac{1}{2}kx^2.$$

The total energy is conserved $\rightarrow$

$$\frac{1}{2}mv_L^2 + \frac{1}{2}kL^2 = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2$$

$$v_L = 0,\ x_0 = 0$$

$$\frac{1}{2}kL^2 = \frac{1}{2}mv_0^2 \Rightarrow$$

$$L = v_0\sqrt{\frac{m}{k}} = 4\times\sqrt{\frac{2}{8}} = 1m$$

L is the maximum length of stretching, as well as the amplitude of oscillation.
For question b), you can also use the work energy theorem to find the answer, as easily.

Work = kinetic energy change $\rightarrow$

$$\int_0^L (-kx)dx = 0 - \frac{1}{2}mv_0^2 \Rightarrow$$

$$-\frac{1}{2}kL^2 = -\frac{1}{2}mv_0^2 \Rightarrow$$

$$L = v_0\sqrt{\frac{m}{k}}$$

c) We need to find the position function x(t). Here we have to go back to the F = ma which gives us the equation of motion →

$$m\frac{d^2x}{dt^2} = F = -kx$$

which is

$$\ddot{x} + \frac{k}{m}x = 0 \qquad (1)$$

We can recognize the equation is a homogeneous second order ODE with constant coefficients and it has a general solution of the form

$$x(t) = A\sin(wt - q_0) \qquad (2)$$

Substituting (2) into (1), noting

$$\ddot{x}(t) = -Aw^2\sin(wt - q_0)$$

(1) becomes

$$-Aw^2\sin(wt - q_0) + \frac{k}{m}A\sin(wt - q_0) = 0 \rightarrow$$

$$\left(-w^2 + \frac{k}{m}\right)A\sin(wt - q_0) = 0 \qquad (3)$$

(3) requires that $-w^2 + \frac{k}{m} = 0$ therefore

$$w = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{2}} = 2s^{-1} \qquad (4)$$

ω has to have the above value for (2) to satisfy (1) - ω is the intrinsic frequency of the given oscillator. Just to satisfy (1), A and θ_0 in (2) can be anything. For our particular problem, we need to determine A and θ_0. From question b), we already know that the amplitude is L, therefore A = L = 1 m (you can also determine A from constant v_0=2 m/s at t=0). The other condition is x=0 at t=0 →

$$0 = \sin(\omega 0 - \theta_0) \rightarrow \theta_0 = 0$$

so the position function is

$$x(t) = \sin(\omega\tau) = \sin(2t).$$

d) $v(t) = dx/dt = \omega \cos(\omega t)$.

e) at t =200s,

$x(200) = \sin(400.0)$ = (use you calculator – here 400 is in radian) m
$v(200) = 2 \cos(400.0)$ m/s

f) $K(200) = \frac{1}{2} mv^2(400) = \frac{1}{2} \times 2 \times 4 \times \sin^2(400) = 4\sin^2(400)$

g) $V(200) = kx^2/2 = 8 \times \cos^2(400.0)/2 = 4\cos^2(400)$

h) $E(200) = k+V = 4 (\sin^2(400) + \cos^2(400)] = 4$ kg m^2 s^{-2}

$$E(0) = \frac{1}{2} mv^2(0) = \frac{1}{2} \times 2 \times 4 = 4 \text{ kg m}^2 \text{ s}^{-2}$$

therefore the total energy is conserved from t =0 to t=200s.