Since the atmosphere is a relatively thin layer enveloping the earth, the radius r may be replaced with little error by the mean radius at sea level, a, and in the velocity components (1-13) as well. However, in order to maintain the *angular momentum* principle, as noted by Phillips (1966), the terms involving w on the right sides of (1-14) must be omitted. The result is an approximate form of (1-15) with r replaced by a. In addition, certain terms in the last equation are omitted for consistency with respect to equivalent vector invariant form (1-11). The resulting approximate equations are:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \left(f + \frac{u\tan\varphi}{a}\right)v + F_\lambda \tag{1-16}$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \left(f + \frac{u\tan\varphi}{a}\right)u + F_\varphi \tag{1-17}$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z \tag{1-18}$$

Here the vertical component of the earth's vorticity, $f = 2\Omega\sin\varphi$, is called the coriolis parameter. The angular momentum principle corresponding to the last set of equations is just (1-15) with r replaced by a, which is a consistent approximation.

When the vertical scale of the disturbances is much smaller than the horizontal scale, the vertical acceleration dw/dt may be neglected; and (1-18) without friction reduces to the *hydrostatic equation*

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{1-20}$$

The thermodynamic equation 1-9b involves total derivatives that can be readily expanded in any coordinate system (A-21) so (1-9b) will suffice for the present.

1-8 MAP PROJECTIONS

For various purposes—analysis, prediction, and depiction of the meteorological variables—it is useful to map all or part of the surface of the earth on a plane. Such map projections should be as nearly like the spherical surface as possible, but it is obvious that some features will be lost. It is very important to preserve the angle between intersecting curves, for example the right angles between latitude circles and meridians. Maps possessing this valuable property are called *conformal*. If distances were preserved from sphere to projection, the map would be termed *isometric*. While this feature is not maintained among the maps used by meteorologists, the distortion of distance can be kept to a tolerable level.

Three commonly used maps are the *polar stereographic*, which is convenient for mapping a hemisphere or a bit more; the *Mercator cylindrical* projection which is especially good for an equatorial band, and the *Lambert conical*

projection, all of which are shown in Figure 1-2. The polar stereographic and Mercator maps are special cases of the Lambert conical projection. When the spherical surface has been mapped onto a plane it may be desirable to set up a new system of coordinates on the plane and further transform the equations of motion into the map coordinates.

1-8-1 Polar Stereographic Projection

This projection has been widely used in numerical weather prediction. The mapping from sphere to plane is accomplished by the transformation

$$r = 2a\tan(\phi/2) \quad \text{and} \quad \theta = \lambda \tag{1-21}$$

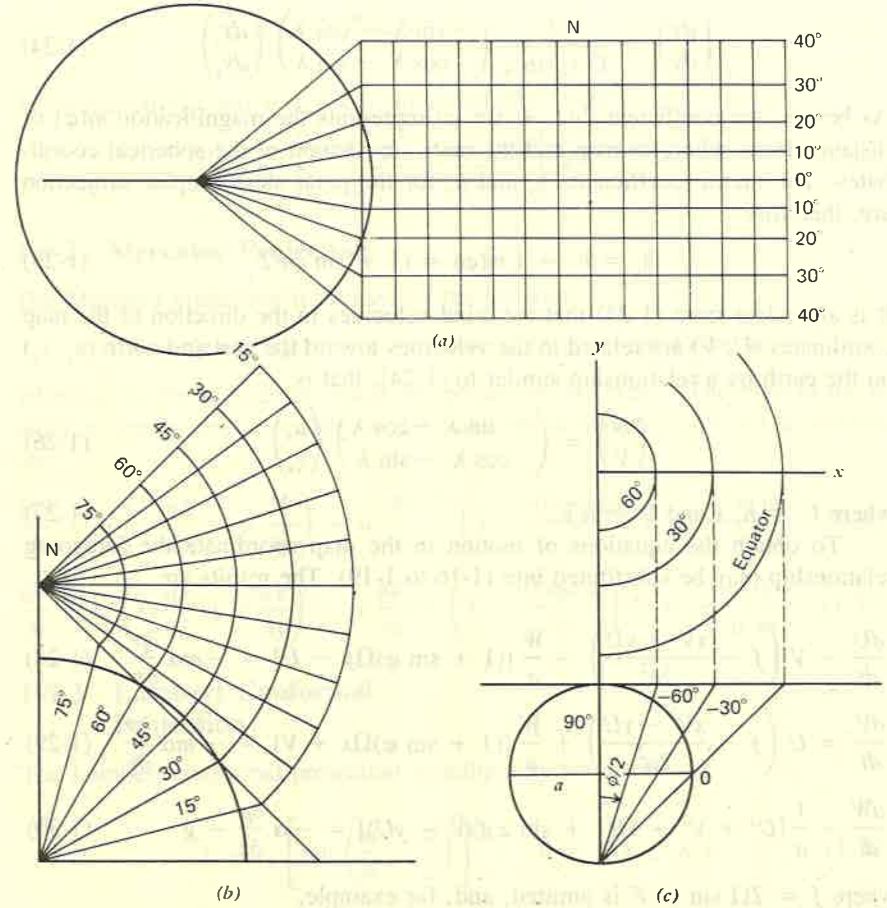

Figure 1-2. (*a*) Mercator cylindrical projection true at 20° latitude, (*b*) Lambert conical projection true at 30° and 60° latitude, (*c*) polar stereographic projection true at 90°.

The first formula follows immediately from Figure 1-2. The symbol a represents the mean earth radius and r is the radius of a latitude circle on the map with colatitude ϕ with $\phi = \pi/2 - \varphi$; (1-21) can be written as

$$r = a\, m(\varphi)\cos\varphi \qquad \theta = \lambda \qquad m(\varphi) = 2/(1 + \sin\varphi) \tag{1-22}$$

where $m(\varphi)$ is the image scale.

Cartesian coordinates may be introduced on the map with $x = r\cos\theta$, $y = r\sin\theta$, which can be written in terms of φ and λ

$$x = \frac{2a\cos\varphi\cos\lambda}{1 + \sin\varphi} \qquad y = \frac{2a\cos\varphi\sin\lambda}{1 + \sin\varphi} \qquad z = z \tag{1-23}$$

Differentiating these expressions leads to relationships between map distances and distance on the sphere (dx_s, dy_s), which may be put in a matrix form:

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \frac{2}{1 + \sin\varphi} \begin{pmatrix} -\sin\lambda & -\cos\lambda \\ \cos\lambda & -\sin\lambda \end{pmatrix} \begin{pmatrix} dx_s \\ dy_s \end{pmatrix} \tag{1-24}$$

As before, the coefficient $2/(1 + \sin\varphi)$ represents the magnification $m(\varphi)$ of distance from sphere to map and the matrix a rotation of the spherical coordinates. The metric coefficients h_x and h_y for the polar stereographic projection are, therefore

$$h_x = h_y = 1/m(\varphi) = (1 + \sin\varphi)/2 \tag{1-25}$$

It is also clear from (1-24) that the wind velocities in the direction of the map coordinates (U, V) are related to the velocities toward the east and north (u_s, v_s) on the earth by a relationship similar to (1-24), that is,

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} -\sin\lambda & -\cos\lambda \\ \cos\lambda & -\sin\lambda \end{pmatrix} \begin{pmatrix} u_s \\ v_s \end{pmatrix} \tag{1-26}$$

where $U = h_x\dot{x}$ and $V = h_y\dot{y}$ (1-27)

To obtain the equations of motion in the map coordinate the foregoing relationship may be substituted into (1-16 to 1-19). The results are

$$\frac{dU}{dt} - V\left(f - \frac{xV - yU}{2a^2}\right) - \frac{W}{a}[(1 + \sin\varphi)\Omega y - U] = -m\alpha\frac{\partial p}{\partial x} \tag{1-28}$$

$$\frac{dV}{dt} = U\left(f - \frac{xV - yU}{2a^2}\right) + \frac{W}{a}[(1 + \sin\varphi)\Omega x + V] = -m\alpha\frac{\partial p}{\partial y} \tag{1-29}$$

$$\frac{dW}{dt} - \frac{1}{a}[U^2 + V^2 + \Omega(1 + \sin\varphi)(xV - yU)] = -\alpha\frac{\partial p}{\partial z} - g \tag{1-30}$$

where $f = 2\Omega\sin\varphi$, F is omitted, and, for example,

$$dU/dt = \partial U/\partial t + m\left(U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right) + W\frac{\partial U}{\partial z} \tag{1-31}$$

When the hydrostatic approximation is used in place of (1-30), the terms explicitly involving W on the left in the horizontal equations (1-28 to 1-29) must be omitted for they exactly cancel all the terms, except dW/dt, on the left side of (1-30) when a kinetic energy equation is formed,

$$d(U^2 + V^2 + W^2)/dt = \ldots.$$

Equations 1-28 to 1-31 then reduce to a form similar to (1-16) to (1-18)

$$\frac{dU}{dt} - V\left(f - \frac{xV - yU}{2a^2}\right) = -m\alpha\frac{\partial p}{\partial x} \tag{1-32}$$

$$\frac{dV}{dt} + U\left(f - \frac{xV - yU}{2a^2}\right) = -m\alpha\frac{\partial p}{\partial y} \tag{1-33}$$

$$0 = \frac{1}{\rho}\frac{\partial p}{\partial z} \quad g \tag{1-34}$$

The equation of continuity needs little alteration, that is,

$$\frac{\partial\rho}{\partial t} + m^2\left[\frac{\partial(\rho U/m)}{\partial x} + \frac{\partial(\rho V/m)}{\partial y}\right] + \frac{\partial(\rho W)}{\partial z} = 0 \tag{1-35}$$

1-8-2 Mercator Projection

The Mercator projection is defined by the equation

$$x = (a\cos\varphi_0)\lambda \qquad y = a\cos\varphi_0\,\ell n[(1 + \sin\varphi)/\cos\varphi]$$

where φ_0 is the latitude at which the projection is "true." The map factor is $m(\varphi) = \cos\varphi_0/\cos\varphi = 1/h_x = 1/h_y$; and the horizontal equations of motion are

$$\frac{\partial U}{\partial t} + m\left(U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right) + w\frac{\partial U}{\partial z} - \left(f + \frac{U\tan\varphi}{a}\right)V = -\frac{m}{\rho}\frac{\partial p}{\partial x} \tag{1-36}$$

$$\frac{\partial V}{\partial t} + m\left(U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y}\right) + w\frac{\partial V}{\partial z} + \left(f + \frac{U\tan\varphi}{a}\right)U = -\frac{m}{\rho}\frac{\partial p}{\partial y} \tag{1-37}$$

1-8-3 Lambert Conformal Projection

The Lambert conformal projection is defined by the equations:

$$r = r_0\left[\tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)\right]^K \qquad \theta = K(\lambda - \lambda_0) \tag{1-38}$$

The two constants r_0 and K may be chosen to make the projection "true" at latitudes φ_1 and φ_2. When this is done (1-38) may be written as

$$r = (a/K)\, m(\varphi)\cos\varphi \qquad \theta = K(\lambda - \lambda_0) \tag{1-39}$$

where

$$m(\varphi) = \left(\frac{\cos\varphi}{\cos\varphi_1}\right)^{(K-1)} \left(\frac{1+\sin\varphi_1}{1+\sin\varphi}\right)^{K}$$

$$K = \ell n\left(\frac{\cos\varphi_1}{\cos\varphi_2}\right) \div \ell n\left\{\frac{\tan[(\pi/4)-(\varphi_1/2)]}{\tan[(\pi/4)-\varphi_2/2)]}\right\} \qquad (1\text{-}40)$$

The equations of motion are identical in form to (1-32) to (1-34).

1-8-4 Additional Remarks

Global integrations of the prediction equations have been made for climatological studies for about a decade and also some global forecasts have been made on a limited basis for both short and extended ranges. As computer power increases, multilevel global forecasts on a 100 to 200 km mesh will be made on a routine basis. Global models have the advantage that no lateral boundary conditions have to be imposed, whereas limited areas require a treatment of the lateral boundaries. The global forecasts will be supplemented by limited area fine mesh models to provide sufficient resolution for the shorter synoptic scales and mesoscale phenomena.

1-9 ALTERNATE VERTICAL COORDINATES

The height z (or radial distance r) is not the most convenient vertical coordinate for many purposes. Other vertical coordinates that have been used with advantage are: pressure, p; $\ell n p/p_0$; pressure normalized with surface pressure, $\sigma = p/p_s$; and potential temperature, θ.

Consider a generalized vertical coordinate ζ [Kasahara (1974)], which is assumed to be related to the height z by a single-valued monotonic function. In terms of the z coordinate, ζ is a function of x, y, z, and t, that is, $\zeta = \zeta(x,y,z,t)$. On the other hand, in terms of ζ as a vertical coordinate, z becomes a dependent variable, so that $z = z(x,y,\zeta,t)$. Any other scalar (or vector) dependent variable, say A, may be expressed in terms of either coordinate system as $A(x,y,z,t)$ or $A(x,y,\zeta,t)$. These functions become identical when either z or σ is replaced by its functional form in terms of the other, that is,

$$A(x,y,\zeta,t) \equiv A(x,y,z(x,y,\zeta,t),t)$$

Now if a partial derivative is taken with respect to s where s is x, y, or t, the result is

$$\left(\frac{\partial A}{\partial s}\right)_\zeta = \left(\frac{\partial A}{\partial s}\right)_z + \frac{\partial A}{\partial z}\left(\frac{\partial z}{\partial s}\right)_\zeta \qquad s = x,y, \text{ or } t \qquad (1\text{-}41)$$

where the subscript ζ or z denotes the particular vertical coordinate. Similarly, the vertical derivatives are related as follows:

$$\frac{\partial A}{\partial \zeta} = \frac{\partial A}{\partial z}\frac{\partial z}{\partial \zeta} \text{ or, alternatively, } \frac{\partial A}{\partial z} = \frac{\partial A}{\partial \zeta}\frac{\partial \zeta}{\partial z} \qquad (1\text{-}42)$$

If (1-42) is substituted into (1-41), the result is

$$\left(\frac{\partial A}{\partial s}\right)_\zeta = \left(\frac{\partial A}{\partial s}\right)_z + \frac{\partial A}{\partial \zeta}\frac{\partial \zeta}{\partial z}\left(\frac{\partial z}{\partial s}\right)_\zeta \qquad (1\text{-}43)$$

The last expression may be used successively with $s = x$ and y to form the gradient of A and similarly with the components of a vector **B** to give the two-dimensional divergence with the results,

$$\nabla_\zeta A = \nabla_z A + \frac{\partial A}{\partial \zeta}\frac{\partial \zeta}{\partial z}\nabla_\zeta z \qquad (1\text{-}44)$$

$$\nabla_\zeta \cdot \mathbf{B} = \nabla_z \cdot \mathbf{B} + \frac{\partial \mathbf{B}}{\partial \zeta}\frac{\partial \zeta}{\partial z}\cdot\nabla_\zeta z \qquad (1\text{-}45)$$

When $s = t$, the result is

$$\left(\frac{\partial A}{\partial t}\right)_\zeta = \left(\frac{\partial A}{\partial t}\right)_z + \frac{\partial A}{\partial \zeta}\frac{\partial \zeta}{\partial z}\left(\frac{\partial z}{\partial t}\right)_\zeta \qquad (1\text{-}46)$$

The total derivative in ζ-coordinates is

$$\frac{dA}{dt} = \left(\frac{\partial A}{\partial t}\right)_\zeta + \mathbf{V}\cdot\nabla_\zeta A + \dot{\zeta}\frac{\partial A}{\partial \zeta} \qquad (1\text{-}47)$$

where **V** is the horizontal velocity henceforth, $\dot{\zeta} = d\zeta/dt$ is the *vertical velocity* in the ζ system and A can be a scalar or a vector (assuming the metric terms are taken care of).

The foregoing expressions may be used to transform the equations of motion from the z-coordinate into the ζ-coordinate. By virtue of (1-44) the horizontal pressure force transforms as follows

$$-\alpha\nabla_z p = -\alpha\nabla_\zeta p + \alpha\frac{\partial p}{\partial z}\nabla_\zeta z = -\alpha\nabla_\zeta p - \nabla_\zeta \phi \qquad (1\text{-}48)$$

where $\phi = gz$ is the geopotential.

Thus the horizontal equation of frictionless motion in ζ coordinates becomes

$$\frac{d\mathbf{V}}{dt} = -\alpha\nabla_\zeta p - \nabla_\zeta\phi - f\mathbf{k}\times\mathbf{V} + \mathbf{F} \qquad (1\text{-}49)$$

Using (1-42) the *hydrostatic equation*, $\alpha(\partial p/\partial z) + g = 0$, becomes

$$\alpha\frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial z} + g = 0 \qquad \text{or} \qquad \alpha\frac{\partial p}{\partial \zeta} + \frac{\partial \phi}{\partial \zeta} = 0 \qquad (1\text{-}50)$$