3 Application of the Governing Equations to Turbulent Flow

To quantitatively describe and forecast the state of the boundary layer, we turn to the equations of fluid mechanics that describe the dynamics and thermodynamics of the gases in our atmosphere. Motions in the boundary layer are slow enough compared to the speed of light that the Galilean/Newtonian paradigm of classical physics applies. These equations, collectively known as the ***equations of motion***, contain time and space derivatives that require initial and boundary conditions for their solution.

Although the equations of motion together with other conservation equations can be applied directly to turbulent flows, rarely do we have sufficient initial and boundary condition information to resolve all turbulent scales down to the smallest eddy. We often don't even care to forecast all eddy motions. For simplicity, we instead pick some cut-off eddy size below which we include only the statistical effects of turbulence. In some mesoscale and synoptic models the cutoff is on the order of 10 to 100 km, while for some boundary layer models known as ***large eddy simulation models*** the cutoff is on the order of 100 m.

The complete set of equations as applied to the boundary layer are so complex that no analytical solution is known. As in other branches of meteorology, we are forced to find approximate solutions. We do this by either finding exact analytical solutions to simplified subsets of the equations, or by finding approximate numerical solutions to a more complete set of equations. Both approximations are frequently combined to allow boundary layer meteorologists to study particular phenomena.

In this chapter we start with the basic governing equations and statistically average over the smaller eddy sizes. Along the way we demonstrate simplifications based on boundary layer scaling arguments. Numerical methods for solving the resulting set of equations are not covered.

3.1 Methodology

Because the upcoming derivations are sometimes long and involved, it is easy "to lose sight of the forest for the trees". The following summary gives the steps that will be taken in the succeeding sections to develop prognostic equations for mean quantities such as temperature and wind:

Step 1. Identify the basic governing equations that apply to the boundary layer.
Step 2. Expand the total derivatives into the local and advective contributions.
Step 3. Expand dependent variables within those equations into mean and turbulent (perturbation) parts.
Step 4. Apply Reynolds averaging to get the equations for mean variables within a turbulent flow.
Step 5. Add the continuity equation to put the result into flux form.

Additional steps take us further towards understanding the nature of turbulence itself:

Step 6. Subtract the equations of step 5 from the corresponding ones of step 3 to get equations for the turbulent departures from the mean.
Step 7. Multiply the results of step 6 by other turbulent quantities and Reynolds average to yield prognostic equations for turbulence statistics such as kinematic flux or turbulence kinetic energy.

Section 3.2 covers steps 1 and 2. Section 3.3 takes a side road to look at some simplifications and scaling arguments. In section 3.4 we get back on track and utilize steps 3-5 to derive the desired prognostic equations. After a few more simplifications in section 3.5, a summary of the governing equations for mean variables in turbulent flow is presented.

Steps 6 and 7 are addressed in Chapters 4 and 5.

3.2 Basic Governing Equations

Five equations form the foundation of boundary layer meteorology: the equation of state, and the conservation equations for mass, momentum, moisture, and heat. Additional equations for scalar quantities such as pollutant concentration may be added. It is assumed that the reader has already been exposed to these equations; hence, the derivations are not given here.

3.2.1 Equation of State (Ideal Gas Law)

The ideal gas law adequately describes the state of gases in the boundary layer:

$$p = \rho_{air} \, \Re \, T_v \tag{3.2.1}$$

where p is pressure, ρ_{air} is the density of moist air, T_v is the virtual absolute temperature, and $\Re$ is the gas constant for **dry** air ($\Re$ = 287 J·K^{-1} kg^{-1}). Sometimes, the density of moist air is abbreviated as ρ for simplicity.

3.2.2 Conservation of Mass (Continuity Equation)

Two equivalent forms of the continuity equation are

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U_j)}{\partial x_j} = 0 \qquad (3.2.2a)$$

and

$$\frac{d\rho}{dt} + \rho \frac{\partial U_j}{\partial x_j} = 0 \qquad (3.2.2b)$$

where the definition of the total derivative is used to convert between these forms.

If $\mathbb{V}$ and $\mathbb{L}$ are typical velocity and length scales for the boundary layer, then it can be shown (Businger, 1982) that $(d\rho/dt)/\rho \ll \partial U_j/\partial x_j$ if the following conditions are met: (1) $\mathbb{V} \ll 100$ m/s; (2) $\mathbb{L} \ll 12$ km; (3) $\mathbb{L} \ll C_s^2/g$; and (4) $\mathbb{L} \ll C_s/f$, where C_s is the speed of sound and f is frequency of any pressure waves that might occur. Since these conditions are generally met for all turbulent motions smaller than mesoscale, (3.2.2b) reduces to

$$\frac{\partial U_j}{\partial x_j} = 0 \qquad (3.2.2c)$$

This is the ***incompressibility*** approximation.

3.2.3 Conservation of Momentum (Newton's Second Law)

As presented at the end of section 2.8.2, one form for the momentum equation is

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\delta_{i3}\, g - 2\,\varepsilon_{ijk}\,\Omega_j\, U_k - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} \qquad (3.2.3a)$$

I II III IV V VI

Term I represents storage of momentum (inertia).
Term II describes advection.
Term III allows gravity to act vertically.
Term IV describes the influence of the earth's rotation (Coriolis effects).
Term V describes pressure-gradient forces.
Term VI represents the influence of viscous stress.

In term IV, the components of the angular velocity vector of the earth's rotation Ω_j are $[0, \omega \cos(\phi), \omega \sin(\phi)]$ where ϕ is latitude and $\omega = 2\pi$ radians/24h $= (7.27\text{x}10^{-5}\ \text{s}^{-1})$ is the angular velocity of the earth. Often term IV is written as $+ f_c\, \varepsilon_{ij3}\, U_j$, where the ***Coriolis parameter*** is defined as $f_c = 2\, \omega \sin \phi \;=\; (1.45\text{x}10^{-4}\ \text{s}^{-1}) \sin \phi$. For a latitude of about 44° (e.g., southern Wisconsin), $f_c = 10^{-4}\ \text{s}^{-1}$.

To a close approximation, air in the atmosphere behaves like a Newtonian fluid. Thus, the expression for viscous stress from section 2.9.3 allows us to write term VI as:

$$\text{Term VI} = \left(\frac{1}{\rho}\right)\frac{\partial}{\partial x_j}\left\{\mu\left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right] - \left(\frac{2}{3}\right)\mu\left[\frac{\partial U_k}{\partial x_k}\right]\delta_{ij}\right\}$$

where the bulk viscosity coefficient μ_B was assumed to be near zero. Upon applying the derivative to each term, assuming that the viscosity μ is not a function of position, and rearranging, this expression can be written as:

$$\text{Term VI} = \left(\frac{\mu}{\rho}\right)\left\{\frac{\partial^2 U_i}{\partial x_j^2} + \frac{\partial}{\partial x_i}\left[\frac{\partial U_j}{\partial x_j}\right] - \left(\frac{2}{3}\right)\frac{\partial}{\partial x_i}\left[\frac{\partial U_k}{\partial x_k}\right]\right\}$$

By assuming incompressibility, this reduces to

$$\text{Term VI} = \nu \frac{\partial^2 U_i}{\partial x_j^2}$$

where the kinematic viscosity, ν, has been substituted for μ/ρ.

Substituting this back into (3.2.3a) gives the form for the momentum equation that is most often used as a starting point for turbulence derivations:

$$\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x_j} = -\delta_{i3}\, g + f_c\, \varepsilon_{ij3}\, U_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 U_i}{\partial x_j^2} \qquad (3.2.3b)$$

I II III IV V VI

where each term represents the same process as before.

3.2.4 Conservation of Moisture

Let q_T be the total specific humidity of air; namely, the mass of water (all phases) per unit mass of moist air. The conservation of water substance can be written, assuming incompressibility, as

$$\underset{\text{I}}{\frac{\partial q_T}{\partial t}} + \underset{\text{II}}{U_j \frac{\partial q_T}{\partial x_j}} = \underset{\text{VI}}{\nu_q \frac{\partial^2 q}{\partial x_j^2}} + \underset{\text{VII}}{\frac{S_{q_T}}{\rho_{air}}} \tag{3.2.4a}$$

where ν_q is the molecular diffusivity for water vapor in the air. S_{q_T} is a net moisture source term (sources - sinks) for the remaining processes not already included in the equation. Its units are: mass of total water per unit volume per unit time.

By splitting the total humidity into vapor (q) and non-vapor (q_L) parts using $q_T = q + q_L$ and $S_{q_T} = S_q + S_{q_L}$, (3.2.4a) can be rewritten as a pair of coupled equations

$$\frac{\partial q}{\partial t} + U_j \frac{\partial q}{\partial x_j} = \nu_q \frac{\partial^2 q}{\partial x_j^2} + \frac{S_q}{\rho_{air}} + \frac{E}{\rho_{air}} \tag{3.2.4b}$$

and

$$\underset{\text{I}}{\frac{\partial q_L}{\partial t}} + \underset{\text{II}}{U_j \frac{\partial q_L}{\partial x_j}} = \underset{\text{VI}}{} + \underset{\text{VII}}{\frac{S_{q_L}}{\rho_{air}}} - \underset{\text{VIII}}{\frac{E}{\rho_{air}}} \tag{3.2.4c}$$

where E represents the mass of water vapor per unit volume per unit time being created by a phase change from liquid or solid. The convergence of falling liquid or solid water (e.g., precipitation) that is not advecting with the wind is included as part of term VII. It has been assumed in (3.2.4c) that molecular diffusion has a negligible effect on liquid and solid precipitation or cloud particles.

Terms I, II, and VI are analogous to the corresponding terms in the momentum equation. Term VII is a net body source term, and term VIII represents the conversion of solid or liquid into vapor.

3.2.5 Conservation of Heat (First Law of Thermodynamics)

The First Law of Thermodynamics describes the conservation of enthalpy, which includes contributions from both sensible and latent heat transport. In other words, the water vapor in air not only transports sensible heat associated with its temperature, but it has the potential to release or absorb additional latent heat during any phase changes that might occur. To simplify the equations describing enthalpy conservation, micrometeorologists often utilize the phase change information, E, contained in the moisture conservation equations. Thus, an equation for θ can be written

$$\underset{\text{I}}{\frac{\partial \theta}{\partial t}} + \underset{\text{II}}{U_j \frac{\partial \theta}{\partial x_j}} = \underset{\text{VI}}{\nu_\theta \frac{\partial^2 \theta}{\partial x_j^2}} - \underset{\text{VII}}{\frac{1}{\rho C_p}\left(\frac{\partial Q_j^*}{\partial x_j}\right)} - \underset{\text{VIII}}{\frac{L_p E}{\rho C_p}} \qquad (3.2.5)$$

where ν_θ is the thermal diffusivity, and L_p is the latent heat associated with the phase change of E. The values for latent heat at 0°C are $L_v = 2.50 \times 10^6$ J/kg (gas:liquid), $L_f = 3.34 \times 10^5$ J/kg (liquid:solid), and $L_s = 2.83 \times 10^6$ J/kg of water (gas:solid).

Q^*_j is the component of net radiation in the jth direction. The specific heat for ***moist*** air at constant pressure, C_p, is approximately related to the specific heat for dry air, $C_{pd} = 1004.67$ J kg^{-1} K^{-1}, by $C_p = C_{pd}(1 + 0.84\,q)$. Given typical magnitudes of q in the boundary layer, it is important not to neglect the moisture contribution to C_p.

Terms I, II, and VI are the storage, advection, and molecular diffusion terms, as before. Term VII is the "body source" term associated with radiation divergence. Term VIII is also a "body source" term associated with latent heat released during phase changes. These body source terms affect the whole volume, not just the boundaries.

3.2.6 Conservation of a Scalar Quantity

Let C be the concentration (mass per volume) of a scalar such as a tracer in the atmosphere. The conservation of tracer mass requires that

$$\underset{\text{I}}{\frac{\partial C}{\partial t}} + \underset{\text{II}}{U_j \frac{\partial C}{\partial x_j}} = \underset{\text{VI}}{\nu_C \frac{\partial^2 C}{\partial x_j^2}} + \underset{\text{VII}}{S_C} \qquad (3.2.6)$$

where ν_C is the molecular diffusivity of constituent C. S_C is the body source term for the remaining processes not already in the equation, such as chemical reactions. The physical interpretation of each term is analogous to that of (3.2.4c).

3.3 Simplifications, Approximations, and Scaling Arguments

Under certain conditions the magnitudes of some of the terms in the governing equations become smaller than the other terms and can be neglected. For these situations the equations become simpler — a fact that has allowed advances to be made in atmospheric dynamics that would otherwise have been more difficult or impossible.

One simplification is called the ***shallow motion approximation*** (Mahrt, 1986). This approximation is valid if all of the following conditions are true:

1) the vertical depth scale of density variations in the boundary layer is much shallower than the scale depth of the lower atmosphere. (This latter scale depth $= \rho\, (\partial\rho/\partial z)^{-1} \cong 8$ km.);
2) advection and divergence of mass at a fixed point approximately balance, leaving only slow or zero variations of density with time.
3) the perturbation magnitudes of density, temperature, and pressure are much less than their respective mean values; and

A more stringent simplification, called the ***shallow convection approximation***, requires all of the conditions above plus:

4) the mean lapse rate $(\partial T/\partial z)$ can be negative, zero, or even slightly positive. For the statically stable positive case, $(\partial T/\partial z) \ll g/\Re$, where $g/\Re = 0.0345$ K/m; and
5) the magnitude of the vertical perturbation pressure gradient term must be of the same order or less than the magnitude of the buoyancy term in the equation of motion.

This latter condition says that vertical motion is limited by buoyancy, which is origin of the term "shallow convection".

We have already employed conditions (1) and (2) to yield the incompressible form of the continuity equation. The other conditions will be applied below to yield further simplifications.

3.3.1 Equation of State

Start with the equation of state (3.2.1) and split the variables into mean and turbulent parts: $\rho = \overline{\rho} + \rho'$, $T_v = \overline{T_v} + T_v'$, $p = \overline{P} + p'$. The result can be rearranged to be

$$\frac{\overline{P}}{\Re} + \frac{p'}{\Re} = (\overline{\rho} + \rho')\cdot(\overline{T_v} + T_v')$$

or

$$\frac{\overline{P}}{\Re} + \frac{p'}{\Re} = \overline{\rho}\cdot\overline{T_v} + \rho'\,\overline{T_v} + \overline{\rho}\,T_v' + \rho'\,T_v' \qquad (3.3.1a)$$

Upon Reynolds averaging, we are left with

$$\frac{\overline{P}}{\Re} = \overline{\rho}\,\overline{T_v} + \overline{\rho' T_v'}$$

The last term is usually much smaller in magnitude than the others, allowing us to neglect it. As a result, the equation of state holds in the mean:

$$\frac{\overline{P}}{\Re} = \overline{\rho}\,\overline{T_v} \qquad (3.3.1b)$$

This is a reasonable approximation because the equation of state was originally formulated from measurements made with crude, slow-response sensors that were essentially measuring mean quantities. As we shall see in section 3.4, however, we can't make similar assumptions for the other governing equations.

Subtracting (3.3.1b) from (3.3.1a) leaves

$$\frac{p'}{\Re} = \rho'\,\overline{T_v} + \overline{\rho}\,T_v' + \rho' T_v'$$

Finally, dividing by (3.3.1b) gives

$$\frac{p'}{\overline{P}} = \frac{\rho'}{\overline{\rho}} + \frac{T_v'}{\overline{T_v}} + \frac{\rho' T_v'}{\overline{\rho}\,\overline{T_v}}$$

Using condition (3) above and the data below, one can show that the last term is smaller than the others, leaving the ***linearized perturbation ideal gas law***:

$$\frac{p'}{\overline{P}} = \frac{\rho'}{\overline{\rho}} + \frac{T_v'}{\overline{T_v}} \qquad (3.3.1c)$$

Static pressure fluctuations are associated with variations in the mass of air from column to column in the atmosphere. For the larger eddies and thermals in the boundary layer, these fluctuations may be as large as 0.01 kPa (0.1 mb), while for smaller eddies the effect is smaller. Dynamic pressure fluctuations associated with wind speeds of up to about 10 m/s also cause fluctuations of about 0.01 kPa. Thus, for most boundary layer situations, $p'/\overline{P} = 0.01\text{ kPa} / 100\text{ kPa} = 10^{-4}$, which is smaller than $T_v'/\overline{T_v} = 1\text{ K} / 300\text{ K} = 3.33 \times 10^{-3}$. For these cases we can make the shallow convection approximation [conditions (4) & (5)] to neglect the pressure term, yielding:

$$\frac{\rho'}{\overline{\rho}} = -\frac{T_v'}{\overline{T_v}} \qquad (3.3.1d)$$

Using Poisson's relationship with the same scaling as above yields:

$$\frac{\rho'}{\bar{\rho}} = -\frac{\theta_v'}{\bar{\theta}_v} \tag{3.3.1e}$$

Physically, (3.3.1e) states that air that is warmer than average is less dense than average. Although not a surprising conclusion, these equations allow us to substitute temperature fluctuations, easily measurable quantities, in place of density fluctuations, which are not so easily measured.

3.3.2 Flux Form of Advection Terms

All of the conservation equations of section 3.2 include an advection term of the form

$$\text{Advection Term} = U_j \, \partial\xi/\partial x_j$$

where ξ denotes any variable, such as a wind component or humidity. If we multiply the continuity equation (3.2.2c) by ξ, we get $\xi \, \partial U_j/\partial x_j = 0$. Since this term is equal to zero, adding it to the advection term will cause no change (other than the mathematical form). Performing this addition gives

$$\text{Advection Term} = U_j \, \partial\xi/\partial x_j + \xi \, \partial U_j/\partial x_j$$

By using the product rule of calculus, we can combine these two terms to give

$$\text{Advection Term} = \partial(\xi U_j)/\partial x_j \tag{3.3.2}$$

This is called the ***flux form*** of the advection term, because as was demonstrated in section 2.6 the product of $(\xi \, U_j)$ is nothing more than a kinematic flux.

3.3.3 Conservation of Momentum

Vertical Component. By setting $i = 3$ in (3.2.3b), we can focus on just the vertical component of momentum to study the role of gravity, density, and pressure on turbulent motions. Utilizing $U_3 = W$ and the definition of the total derivative, $dU_i/dt = \partial U_i/\partial t + U_j \, \partial U_i/\partial x_j$, gives

$$\frac{dW}{dt} = -g - \frac{1}{\rho}\left(\frac{\partial p}{\partial z}\right) + \nu \frac{\partial^2 W}{\partial x_j^2}$$

In the following development, we will treat viscosity as a constant. Multiply the above equation by ρ and let $\rho = \overline{\rho} + \rho'$, $W = \overline{W} + w'$ and $p = \overline{P} + p'$:

$$(\overline{\rho} + \rho')\frac{d(\overline{W}+w')}{dt} = -(\overline{\rho} + \rho')\,g - \frac{\partial(\overline{P}+p')}{\partial z} + \mu\frac{\partial^2(\overline{W}+w')}{\partial x_j^2}$$

Dividing by $\overline{\rho}$ and rearranging gives:

$$\left(1 + \frac{\rho'}{\overline{\rho}}\right)\frac{d(\overline{W}+w')}{dt} = -\frac{\rho'}{\overline{\rho}}g - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} + \nu\frac{\partial^2(\overline{W}+w')}{\partial x_j^2} - \frac{1}{\overline{\rho}}\left[\frac{\partial\overline{P}}{\partial z} + \overline{\rho}\,g\right]$$

If we assume that the mean state is in ***hydrostatic equilibrium*** ($\partial\overline{P}/\partial z = -\overline{\rho}\,g$), then the term in square brackets is zero. Furthermore, if we remember from section 3.3.1 that $\rho'/\overline{\rho}$ is on the order of 3.33×10^{-3}, then we see that the factor on the left hand side of the equation is approximated by $(1 + \rho'/\overline{\rho}) \cong 1$. We can't neglect, however, the first term on the right hand side of the equal sign, because the product $[\rho'/\overline{\rho}\, g]$ is as large as the other terms in the equation. The process of neglecting density variations in the inertia (storage) term, but retaining it in the buoyancy (gravity) term is called the ***Boussinesq approximation***. These two approximations leave

$$\frac{d(\overline{W}+w')}{dt} = -\frac{\rho'}{\overline{\rho}}g - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} + \nu\frac{\partial^2(\overline{W}+w')}{\partial x_j^2}$$

A prerequisite for the Boussinesq approximation is that the shallow convection conditions be satisfied.

By comparing the original equations to the scaled equations above, one finds differences in the terms involving ρ and g. Thus, a simple way to apply the Boussinesq approximation without performing the complete derivation is stated here:

Practical Application of the Boussinesq Approximation:
Given any of the original governing equations, replace every occurrence of ρ with $\overline{\rho}$, and replace every occurrence of g with $\left[g - (\theta_v'/\overline{\theta_v})g\right]$.

Although subsidence, $\overline{W}$, is important in mass conservation and in the advection of material (moisture, pollutants, etc) from aloft, we see that it is less important in the momentum equation because it is always paired in a linear manner with w'. In fair-weather boundary layers, subsidence can vary from zero to 0.1 m/s, which is considered a relatively large value. This is small compared to the vertical velocity fluctuations, which frequently vary over the range 0 to 5 m/s. Thus, for **only** the momentum equation for fair-weather conditions can we usually ***neglect subsidence:***

$$\overline{W} \cong 0 \tag{3.3.3a}$$

This leaves the vertical component of the momentum equation as

$$\frac{dw'}{dt} = -\left(\frac{\rho'}{\overline{\rho}}\right)g - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} + \nu\frac{\partial^2 w'}{\partial x_j^2}$$

Using (3.3.1) to replace the density variations with temperature variations gives

$$\frac{dw'}{dt} = \left(\frac{\theta_v'}{\overline{\theta_v}}\right)g - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} + \nu\frac{\partial^2 w'}{\partial x_j^2} \tag{3.3.3b}$$

The physical interpretation of the first two terms in (3.3.3b) is that warmer than average air is accelerated upward (i.e., hot air rises). The last two terms describe the influences of pressure gradients and viscous stress on the motion. This equation therefore plays an important role in the evolution of convective thermals.

Horizontal Component. Although the BL winds are rarely geostrophic, we can use the definition of the ***geostrophic wind*** as a substitute variable for the horizontal pressure gradient terms:

$$f_c U_g = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad \text{and} \quad f_c V_g = +\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{3.3.3c}$$

Thus, the horizontal components of (3.2.3b) become

$$\frac{dU}{dt} = -f_c(V_g - V) + \nu \frac{\partial^2 U}{\partial x_j^2} \tag{3.3.3d}$$

$$\frac{dV}{dt} = +f_c(U_g - U) + \nu \frac{\partial^2 V}{\partial x_j^2} \tag{3.3.3e}$$

I II III

Term I is the ***inertia*** or storage term. Term II is sometimes called the ***geostrophic departure*** term, because it is zero when the actual winds are geostrophic. As we stated before, however, the winds are rarely geostrophic in the BL. Term III describes viscous shear stress.

Combined Momentum Equation. Combining the results from the previous two subsections yields

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\varepsilon_{ij3} f_c (U_{gj} - U_j) + \delta_{i3}\left[\frac{\theta_v'}{\overline{\theta_v}} g - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z}\right] + \nu \frac{\partial^2 U_i}{\partial x_j^2} \tag{3.3.3f}$$

where we have applied the shallow convection, incompressibility, hydrostatic and Boussinesq approximations, and where $U_{gj} = (U_g, V_g, 0)$.

3.3.4 Horizontal Homogeneity

Expanding the total derivative of any mean variable, $\overline{\xi}$, yields

$$\frac{d\overline{\xi}}{dt} = \frac{\partial \overline{\xi}}{\partial t} + U\frac{\partial \overline{\xi}}{\partial x} + V\frac{\partial \overline{\xi}}{\partial y} + W\frac{\partial \overline{\xi}}{\partial z} \tag{3.3.4}$$

I II III IV

From examples like Figs 1.12 and 2.9 we saw that averaged variables such as potential temperature or turbulence kinetic energy exhibit large vertical variations over the 1 to 2 km of boundary layer depth. Those same variables, however, usually exhibit a much smaller horizontal variation over the same 1 to 2 km scale. Counteracting this disparity of gradients is a disparity of velocities. Namely, U and V are often on the order of m/s while

W is on the order of mm/s or cm/s. The resulting terms I through IV in the above equation are thus nearly equal in magnitude for many cases.

The bottom line is that we usually can **not** neglect horizontal advection (terms II & III), and we can **not** neglect subsidence (term IV) as it affects the movement of conserved variables.

Sometimes micrometeorologists wish to focus their attention on turbulence effects at the expense of neglecting mean advection. By assuming ***horizontal homogeneity***, we can set $\partial\overline{\xi}/\partial x = 0$ and $\partial\overline{\xi}/\partial y = 0$, and ***neglecting subsidence*** gives $\overline{W} = 0$. Although these assumptions are frequently made by theorists to simplify their derivations, they are rarely valid in the real atmosphere. When they are made, they cause the advection terms of only mean variables (like $\overline{\xi}$) to disappear; the turbulent flux terms do **not** disappear, and in fact are very important.

3.3.5 Reorientating and Rotating the Coordinate System

Although we usually use a Cartesian coordinate system aligned such that the (x, y, z) axes point (east, north, up), sometimes it is convenient to rotate the Cartesian coordinate system about the vertical (z) axes to cause x and y to point in other directions. Some examples include aligning the x-axis with:

the mean wind direction,
the geostrophic wind direction
the direction of surface stress, or
perpendicular to shorelines or mountains.

The only reason for doing this is to simplify some of the terms in the governing equations. For example, by choosing the x-axis aligned with the mean wind, we find U=M and V=0. In such a system, the x-axis is called the ***along-wind direction*** and the y-direction is called the ***crosswind direction***.

3.4 Equations for Mean Variables in a Turbulent Flow

3.4.1 Equation of State

As was already stated in section 3.3.1, the equation of state is assumed to hold in the mean, and is rewritten here for the sake of organization:

$$\frac{\overline{P}}{\Re} = \overline{\rho}\ \overline{T_v} \qquad (3.4.1)$$

3.4.2 Continuity Equation

Start with the continuity equation (3.2.2c) and expand the velocities into mean and turbulent parts to give:

$$\frac{\partial(\overline{U_j} + u_j')}{\partial x_j} = 0$$

or

$$\frac{\partial \overline{U_j}}{\partial x_j} + \frac{\partial u_j'}{\partial x_j} = 0 \qquad (3.4.2a)$$

Next average over time

$$\overline{\frac{\partial \overline{U_j}}{\partial x_j} + \frac{\partial u_j'}{\partial x_j}} = 0$$

Upon applying Reynold's averaging rules, the last term becomes zero, leaving

$$\frac{\partial \overline{U_j}}{\partial x_j} = 0 \qquad (3.4.2b)$$

Thus, the continuity equation holds in the mean. Subtracting this from (3.4.2a) gives the continuity equation for turbulent fluctuations:

$$\frac{\partial u_j'}{\partial x_j} = 0 \qquad (3.4.2c)$$

This equation will allow us to put turbulent advection terms into flux form, in the same manner as was demonstrated for (3.3.2).

3.4.3 Conservation of Momentum

Starting with the conservation of momentum expressed by (3.2.3b), make the Boussinesq approximation:

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\delta_{i3}\left[g - \left(\frac{\theta_v'}{\overline{\theta_v}}\right) g\right] + f_c \varepsilon_{ij3} U_j - \frac{1}{\overline{\rho}} \frac{\partial P}{\partial x_i} + \frac{\nu \partial^2 U_i}{\partial x_j^2}$$

Next, expand the dependent variables into mean and turbulent parts (except for the $\theta_v' / \overline{\theta_v}$ term, for which the expansion has previously been made):

$$\frac{\partial(\overline{U_i} + u_i')}{\partial t} + \frac{(\overline{U_j} + u_j')\partial(\overline{U_i} + u_i')}{\partial x_j} = -\delta_{i3}\left[g - \left(\frac{\theta_v'}{\overline{\theta_v}}\right)g\right] + f_c\varepsilon_{ij3}(\overline{U_j} + u_j')$$

$$- \frac{1}{\overline{\rho}}\frac{\partial(\overline{P}+p')}{\partial x_i} + \nu\partial^2\frac{(\overline{U_i} + u_i')}{\partial x_j^2}$$

Upon performing the indicated multiplications, and separating terms, we find

$$\frac{\partial\overline{U_i}}{\partial t} + \frac{\partial u_i'}{\partial t} + \frac{\overline{U_j}\,\partial\overline{U_i}}{\partial x_j} + \frac{\overline{U_j}\,\partial u_i'}{\partial x_j} + \frac{u_j'\,\partial\overline{U_i}}{\partial x_j} + \frac{u_j'\,\partial u_i'}{\partial x_j} =$$

$$-\delta_{i3}g + \delta_{i3}\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g + f_c\varepsilon_{ij3}\overline{U_j} + f_c\varepsilon_{ij3}u_j' - \frac{1}{\overline{\rho}}\frac{\partial\overline{P}}{\partial x_i} - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_i} + \nu\frac{\partial^2\overline{U_i}}{\partial x_j^2} + \nu\frac{\partial^2 u_i'}{\partial x_j^2} \tag{3.4.3a}$$

Next, average the whole equation:

$$\overline{\frac{\partial\overline{U_i}}{\partial t}} + \overline{\frac{\partial u_i'}{\partial t}} + \overline{\frac{\overline{U_j}\,\partial\overline{U_i}}{\partial x_j}} + \overline{\frac{\overline{U_j}\,\partial u_i'}{\partial x_j}} + \overline{\frac{u_j'\,\partial\overline{U_i}}{\partial x_j}} + \overline{\frac{u_j'\,\partial u_i'}{\partial x_j}} =$$

$$\overline{-\delta_{i3}g} + \overline{\delta_{i3}\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g} + \overline{f_c\varepsilon_{ij3}\overline{U_j}} + \overline{f_c\varepsilon_{ij3}u_j'} - \overline{\frac{1}{\overline{\rho}}\frac{\partial\overline{P}}{\partial x_i}} - \overline{\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_i}} + \overline{\nu\frac{\partial^2\overline{U_i}}{\partial x_j^2}} + \overline{\nu\frac{\partial^2 u_i'}{\partial x_j^2}}$$

By applying Reynolds averaging rules the second, fourth, fifth, eighth, tenth, twelfth and fourteenth terms become zero. We are left with:

$$\frac{\partial\overline{U_i}}{\partial t} + \frac{\overline{U_j}\partial\overline{U_i}}{\partial x_j} + \frac{\overline{u_j'\,\partial u_i'}}{\partial x_j} = -\delta_{i3}g + f_c\varepsilon_{ij3}\overline{U_j} - \frac{1}{\overline{\rho}}\frac{\partial\overline{P}}{\partial x_i} - \frac{\nu\partial^2\overline{U_i}}{\partial x_j^2} \tag{3.4.3b}$$

Finally, multiply the continuity equation for turbulent motions (3.4.2c) by u_i', average it, and add it to (3.4.3b) to put the turbulent advection term into flux form:

$$\frac{\partial \overline{U_i}}{\partial t} + \frac{\overline{U_j} \partial \overline{U_i}}{\partial x_j} + \frac{\partial (\overline{u_i' u_j'})}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} \overline{U_j} - \frac{1}{\overline{\rho}} \frac{\partial \overline{P}}{\partial x_i} + \frac{\nu \partial^2 \overline{U_i}}{\partial x_j^2}$$

By moving this flux term to the right hand side of the equation, we see something very remarkable; namely, the following forecast equation for mean wind is very similar to the basic conservation equation we started with (3.2.3b), except for the addition of the turbulence term at the end.

$$\frac{\partial \overline{U_i}}{\partial t} + \overline{U_j} \frac{\partial \overline{U_i}}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} \overline{U_j} - \frac{1}{\overline{\rho}} \frac{\partial \overline{P}}{\partial x_i} + \frac{\nu \partial^2 \overline{U_i}}{\partial x_j^2} - \frac{\partial (\overline{u_i' u_j'})}{\partial x_j} \tag{3.4.3c}$$

I II III IV V VI X

Term I represents storage of mean momentum (inertia).
Term II describes advection of mean momentum by the mean wind.
Term III allows gravity to act in the vertical direction only.
Term IV describes the influence of the earth's rotation (Coriolis effects).
Term V describes the mean pressure-gradient forces.
Term VI represents the influence of viscous stress on the mean motions.
Term X represents the influence of Reynolds' stress on the mean motions (see section 2.9.2). It can also be described as the divergence of turbulent momentum flux.

Term X can also be written as $(1/\overline{\rho})\ \partial \tau_{ij\ \text{Reynolds}}/\partial x_j$ where $\tau_{ij\ \text{Reynolds}} = -\overline{\rho}\ \overline{u_i'\ u_j'}$. The implication of this last term is that **turbulence must be considered** in making forecasts in the turbulent boundary layer, **even if we are trying to forecast only mean quantities**. Term X can often be as large in magnitude, or larger, than many other terms in the equation. Sometimes term X is labled as "F" by large-scale dynamists to denote friction.

3.4.4 Conservation of Moisture

For total specific humidity, start with (3.2.4a) and split the dependent variables into mean and turbulent parts:

$$\frac{\partial \overline{q_T}}{\partial t} + \frac{\partial q_T'}{\partial t} + \frac{\overline{U_j} \partial \overline{q_T}}{\partial x_j} + \frac{\overline{U_j} \partial q_T'}{\partial x_j} + \frac{u_j' \partial \overline{q_T}}{\partial x_j} + \frac{u_j' \partial q_T'}{\partial x_j} =$$

$$\frac{\nu_q \partial^2 \overline{q}}{\partial x_j^2} + \frac{\nu_q \partial^2 q'}{\partial x_j^2} + \frac{S_{qT}}{\overline{\rho}_{air}} \tag{3.4.4a}$$

where the net remaining source term, S_{qT}, is assumed to be a mean forcing. Next, average the equation, apply Reynolds' averaging rules, and use the turbulent continuity equation to put the turbulent advection term into flux form:

$$\underset{\text{I}}{\frac{\partial \overline{q_T}}{\partial t}} + \underset{\text{II}}{\frac{\overline{U_j} \partial \overline{q_T}}{\partial x_j}} = \underset{\text{VI}}{\frac{\nu_q \partial^2 \overline{q}}{\partial x_j^2}} + \underset{\text{VII}}{\frac{S_{qT}}{\overline{\rho}_{air}}} - \underset{\text{X}}{\frac{\partial(\overline{u_j' q_T'})}{\partial x_j}} \tag{3.4.4b}$$

Term I represents the storage of mean total moisture.
Term II describes the advection of mean total moisture by the mean wind.
Term VI represents the mean molecular diffusion of water vapor.
Term VII is the mean net body source term for additional moisture processes.
Term X represents the divergence of turbulent total moisture flux.

As before, this equation is similar to the basic conservation equation (3.2.4a), except for the addition of the turbulence term at the end. Similar equations can be written for the vapor and non-vapor parts of total specific humidity.

3.4.5 Conservation of Heat

Start with the basic heat conservation equation (3.2.5) and expand the dependent variables into mean and turbulent parts

$$\frac{\partial \overline{\theta}}{\partial t} + \frac{\partial \theta'}{\partial t} + \frac{\overline{U_j} \partial \overline{\theta}}{\partial x_j} + \frac{\overline{U_j} \partial \theta'}{\partial x_j} + \frac{u_j' \partial \overline{\theta}}{\partial x_j} + \frac{u_j' \partial \theta'}{\partial x_j} =$$

$$\frac{\nu_\theta \partial^2 \overline{\theta}}{\partial x_j^2} + \frac{\nu_\theta \partial^2 \theta'}{\partial x_j^2} - \frac{1}{\overline{\rho}\, C_p} \frac{\partial \overline{Q_j^*}}{\partial x_j} - \frac{1}{\overline{\rho}\, C_p} \frac{\partial Q_j^{*\prime}}{\partial x_j} - \frac{L_v E}{\overline{\rho}\, C_p} \tag{3.4.5a}$$

Next, Reynolds average and put the turbulent advection term into flux form to give:

$$\frac{\partial\overline{\theta}}{\partial t} + \frac{\overline{U}_j \partial\overline{\theta}}{\partial x_j} = \frac{\nu_\theta \partial^2\overline{\theta}}{\partial x_j^2} - \frac{1}{\overline{\rho}\, C_p}\frac{\partial \overline{Q_j^*}}{\partial x_j} - \frac{L_v E}{\overline{\rho}\, C_p} - \frac{\partial(\overline{u_j'\theta'})}{\partial x_j} \qquad (3.4.5b)$$

I II VI VII VIII X

Term I represents the mean storage of heat.
Term II describes the advection of heat by the mean wind.
Term VI represents the mean molecular conduction of heat.
Term VII is the mean net body source associated with radiation divergence.
Term VIII is the body source term associated with latent heat release.
Term X represents the divergence of turbulent heat flux.

3.4.6 Conservation of a Scalar Quantity

Start with the basic conservation equation (3.2.6) of tracer C and expand into mean and turbulent parts:

$$\frac{\partial\overline{C}}{\partial t} + \frac{\partial c'}{\partial t} + \frac{\overline{U}_j \partial\overline{C}}{\partial x_j} + \frac{\overline{U}_j \partial c'}{\partial x_j} + \frac{u_j' \partial\overline{C}}{\partial x_j} + \frac{u_j' \partial c'}{\partial x_j} =$$

$$\frac{\nu_c \partial^2\overline{C}}{\partial x_j^2} + \frac{\nu_c \partial^2 c'}{\partial x_j^2} + S_c \qquad (3.4.6a)$$

where the net remaining source term, S_c, is assumed to be a mean forcing. Next, Reynolds average and use the turbulent continuity equation to put the turbulent advection term into flux form:

$$\frac{\partial\overline{C}}{\partial t} + \frac{\overline{U}_j \partial\overline{C}}{\partial x_j} = \frac{\nu_c \partial^2\overline{C}}{\partial x_j^2} + S_c - \frac{\partial(\overline{u_j'c'})}{\partial x_j} \qquad (3.4.6b)$$

I II VI VII X

Term I represents the mean storage of tracer C.
Term II describes the advection of the tracer by the mean wind.
Term VI represents the mean molecular diffusion of the tracer.
Term VII is the mean net body source term for additional tracer processes.
Term X represents the divergence of turbulent tracer flux.

3.5 Summary of Equations, with Simplifications

To simplify usage of the equations, we have collected them in this section and organized them in a way that similarities and differences can be more easily noted. Before we list these equations, however, we can make one additional simplification based on the scale of viscous effects vs. turbulent effects on the mean fields.

3.5.1 The Reynolds Number

The ***Reynolds number***, Re, is defined as

$$\mathrm{Re} \equiv \mathbb{V}\,\mathbb{L}/\nu = \rho\,\mathbb{V}\,\mathbb{L}/\mu \tag{3.5.1}$$

where $\mathbb{V}$ and $\mathbb{L}$ are velocity and length scales in the boundary layer. Given $\nu_{air} \cong 1.5 \times 10^{-5}\ \mathrm{m^2 s^{-1}}$ and the typical scaling values $\mathbb{V} = 5$ m/s and $\mathbb{L} = 100$ m in the surface layer, we find that Re = 3×10^7. In the atmospheric mixed layer, the Reynolds number is even larger. The Reynolds number can be interpreted as the ratio of inertial to viscous forcings.

3.5.2 Neglect of Viscosity for Mean Motions

In each of the conservation equations except mass conservation, there are molecular diffusion/viscosity terms. Observations in the atmosphere indicate that the molecular diffusion terms are several order of magnitudes smaller than the other terms and can be neglected.

For example, after making the hydrostatic assumption, the momentum conservation equation for mean motions in turbulent flow (3.4.3c) can be rewritten as

$$\left[\frac{\partial \overline{U_i}}{\partial t}\right] + \left[\overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j}\right] = \left[f_c\,\varepsilon_{ij3}\,\overline{U_j}\right] - \left[\frac{1}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x}\right] - \left[\frac{1}{\overline{\rho}}\frac{\partial \overline{P}}{\partial y}\right] - \left[\frac{\partial \overline{u_i' u_j'}}{\partial x_j}\right] + \frac{1}{\mathrm{Re}}\left[\mathbb{V}\mathbb{L}\frac{\partial^2 \overline{U_i}}{\partial x_j^2}\right] \tag{3.5.2}$$

Each of the terms in square brackets is roughly the same order of magnitude. The last term, however, is multiplied by (1/Re) -- a very small number (on the order of 10^{-7}). Hence, the last term can be neglected compared to the rest, except in the lowest few centimeters above the surface.

3.5.3 Summary of Equations for Mean Variables in Turbulent Flow

Neglecting molecular diffusion and viscosity, and making the hydrostatic and Boussinesq approximations to the governing equations leaves:

$$\frac{\overline{P}}{\Re} = \overline{\rho}\,\overline{T_v} \tag{3.5.3a}$$

$$\frac{\partial \overline{U_j}}{\partial x_j} = 0 \tag{3.5.3b}$$

$$\frac{\partial \overline{U}}{\partial t} + \overline{U_j}\frac{\partial \overline{U}}{\partial x_j} = -f_c(\overline{V_g} - \overline{V}) - \frac{\partial(\overline{u_j'u'})}{\partial x_j} \tag{3.5.3c}$$

$$\frac{\partial \overline{V}}{\partial t} + \overline{U_j}\frac{\partial \overline{V}}{\partial x_j} = +f_c(\overline{U_g} - \overline{U}) - \frac{\partial(\overline{u_j'v'})}{\partial x_j} \tag{3.5.3d}$$

$$\frac{\partial \overline{q_T}}{\partial t} + \overline{U_j}\frac{\partial \overline{q_T}}{\partial x_j} = +S_{qT}/\overline{\rho_{air}} - \frac{\partial(\overline{u_j'q_T'})}{\partial x_j} \tag{3.5.3e}$$

$$\frac{\partial \overline{\theta}}{\partial t} + \overline{U_j}\frac{\partial \overline{\theta}}{\partial x_j} = -\frac{1}{\overline{\rho}\,C_p}\left[L_v E + \frac{\partial \overline{Q_j^*}}{\partial x_j}\right] - \frac{\partial(\overline{u_j'\theta'})}{\partial x_j} \tag{3.5.3f}$$

$$\frac{\partial \overline{C}}{\partial t} + \overline{U_j}\frac{\partial \overline{C}}{\partial x_j} = +S_c - \frac{\partial(\overline{u_j'c'})}{\partial x_j} \tag{3.5.3g}$$

I II VII X

The similarity between the last five equations reflects that the same forcings are present in each conservation equation:

Term I represents storage.
Term II represents advection.
Term VII represent sundry body forcings.
Term X describes the turbulent flux divergence.

The covariances appearing in term X reinforce the earlier assertion that statistics play an important role in the study of turbulent flow.

In the two momentum equations above, the ***mean geostrophic wind components*** were defined using the mean horizontal pressure gradients:

$$\overline{U_g} = -\frac{1}{f_c \overline{\rho}} \frac{\partial \overline{P}}{\partial y} \quad \text{and} \quad \overline{V_g} = +\frac{1}{f_c \overline{\rho}} \frac{\partial \overline{P}}{\partial x} \tag{3.5.3h}$$

Sometimes the left hand side of equations c thru g are simplified using

$$\frac{d(\,)}{dt} = \frac{\partial(\,)}{\partial t} + \frac{\overline{U_j}\,\partial(\,)}{\partial x_j} \tag{3.5.3i}$$

where the total derivative d( )/dt is inferred to include only ***mean*** advective effects, and not the turbulent effects.

3.5.4 Examples

Many applications will have to wait until more realistic PBL initial and boundary conditions have been covered. For now, just a few artificial sample exercises showing the use of equations (3.5.3) will be presented.

Problem 1. Suppose that the turbulent heat flux decreases linearly with height according to $\overline{w'\theta'} = a - b\,z$, where $a = 0.3$ (K ms^{-1}) and $b = 3\times10^{-4}$ (K s^{-1}). If the initial potential temperature profile is an arbitrary shape (i.e., pick a shape), then what will be the shape of final profile one hour later? Neglect subsidence, radiation, latent heating, and assume horizontal homogeneity.

Solution. Neglecting subsidence, radiation, and latent heating leaves (3.5.3f) as

$$\frac{\partial\overline{\theta}}{\partial t} + \frac{\overline{U}\partial\overline{\theta}}{\partial x} + \frac{\overline{V}\partial\overline{\theta}}{\partial y} = -\frac{\partial(\overline{u'\theta'})}{\partial x} - \frac{\partial(\overline{v'\theta'})}{\partial y} - \frac{\partial(\overline{w'\theta'})}{\partial z}$$

By assuming horizontal homogeneity, the x and y derivatives drop out, giving

$$\frac{\partial\overline{\theta}}{\partial t} = -\frac{\partial(\overline{w'\theta'})}{\partial z}$$

Plugging in the expression for $\overline{w'\theta'}$ gives $\partial\overline{\theta}/\partial t = +b$. This answer is not a function of z; hence, air at each height in the sounding warms at the same rate. Integrating over time from $t = t_o$ to t gives

$$\overline{\theta}\,\big|_T = \overline{\theta}\big|_{to} + b(t - t_0)$$

The warming in one hour is $b\,(t - t_o) = [3\times10^{-4}\ (\text{K/s})]\cdot[\,3600\ (\text{s})] = 1.08$ K.