2 Some Mathematical & Conceptual Tools: Part 1. Statistics

Turbulence is an intrinsic part of the atmospheric boundary layer that must be quantified in order to study it. The randomness of turbulence makes deterministic description difficult. Instead, we are forced to retreat to the use of statistics, where we are limited to average or expected measures of turbulence. In this chapter we review some basic statistical methods and show how measurements of turbulence can be put into a statistical framework. Usually, this involves separating the turbulent from the nonturbulent parts of the flow, followed by averaging to provide the statistical descriptor.

The role of spectra in this separation of parts is also described. Although at first glance we see a labyrinth of motions, turbulence may be idealized as consisting of a variety of different-size swirls or eddies. These eddies behave in a well-ordered manner when displayed in the form of a spectrum.

Statistical descriptors such as the variance or covariance are of limited usefulness unless we can physically interpret them. Variances are shown to be measures of turbulence intensity or turbulence kinetic energy, and covariances are shown to be measures of flux or stress. Flux and stress concepts are explored further, and Einstein's summation notation is introduced as a shorthand way to write these variables.

The concepts developed in this chapter are used extensively in the remainder of the book to help describe the turbulent boundary layer.

2.1 The Signature of Turbulence and Its Spectrum

Since we have lived most of our lives within the turbulent boundary layer, we have developed feelings or intuitions about the nature of turbulence that can be refined to help us classify and describe this phenomenon. Consider figure 2.1 for example, which shows the near-surface wind speed measured during a 2.5 hour period. A number of features stand out.

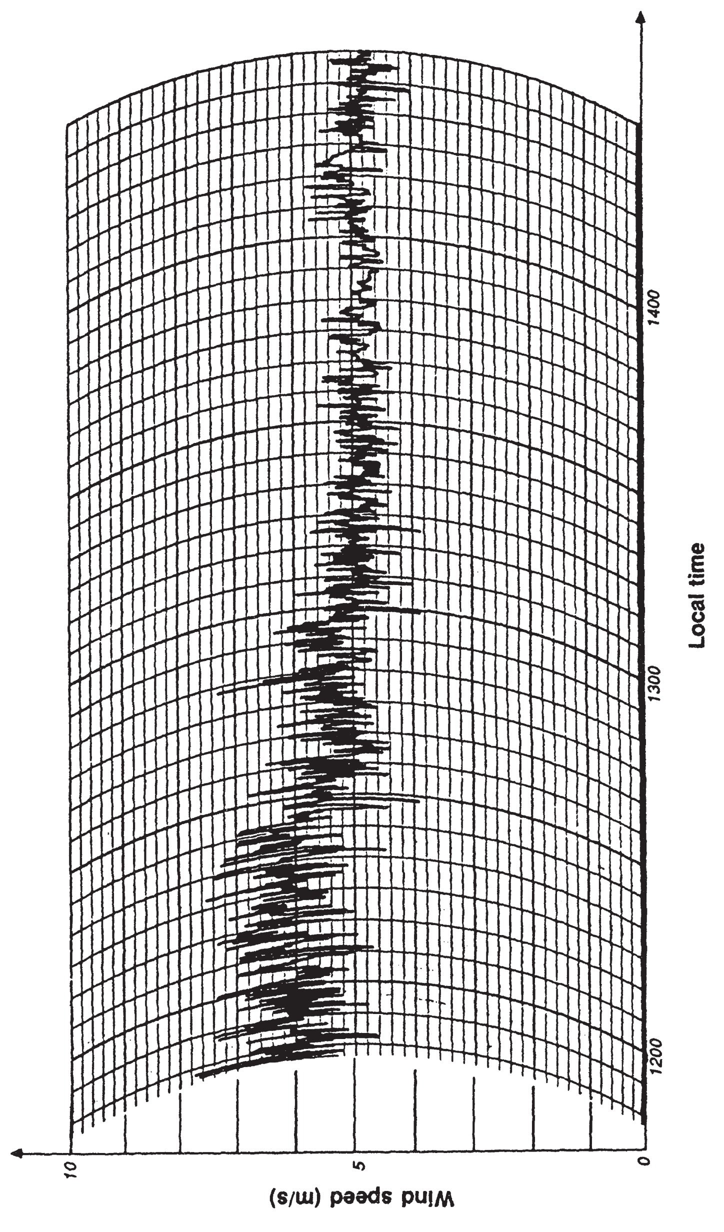


Fig. 2.1 Trace of wind speed obseved in early afternoon.

- The wind speed varies in an irregular pattern — a characteristic signature of turbulence. This quasi-randomness is what makes turbulence different from other motions, like waves.
- We can visually pick out a mean, or typical, value of the wind speed. For example, between noon and 1230 local time the average wind speed is about 6 m/s, while a bit later (between 1400 and 1430) the winds have decreased to about 5 m/s on the average. The ability to find a statistically-stable mean value suggests that turbulence is not completely random.
- The wind speeds do not vary from 0 to 100 m/s in this graph, but rather vary over a limited range of speeds. In other words, there is a measurable and definable intensity to the turbulence that shows up on this graph as the vertical spread of wind speed. The turbulence intensity appears to decrease between noon and 1400 local time.

  Near noon the instantaneous wind speed is often 1 m/s faster or slower than the mean, while at 1400 the wind speed varies by only about 0.5 m/s about its mean. Such a bounded characteristic of the wind speed means that we can use statistics such as the variance or standard deviation to characterize the turbulence intensity.
- There appears to be a wide variety of time-scales of wind variation superimposed on top of each other. If we look closely we see that the time period between each little peak in wind speed is about a minute. The larger peaks seem to happen about every 5 min. There are other variations that indicate a 10 min time period. The smallest detectable variations on this chart are about 10 s long.

  If each of these time variations is associated with a different size turbulent eddy (Taylor's hypothesis. See exercise 5 in Chapter 1), then we can conclude that we are seeing the signature of eddies ranging in size from about 50 m to about 3000 m. In other words, we are observing evidence of the spectrum of turbulence.

The turbulence spectrum is analogous to the spectrum of colors that appears when you shine a light through a prism. White light consists of many colors (i.e., many wavelengths or frequencies) superimposed on one another. The prism is a physical device that separates the colors. We could measure the intensity of each color to learn the magnitude of its contribution to the original light beam. We can perform a similar analysis on a turbulent signal using mathematical rather than physical devices to learn about the contribution of each different size eddy to the total turbulence kinetic energy.

Figure 2.2 shows an example of the spectrum of wind speed measured near the ground. The ordinate is a measure of the portion of turbulence energy that is associated with a particular size eddy. The abscissa gives the eddy size in terms of the time period and frequency of the wind-speed variation. Small eddies have shorter time periods than large eddies (again, using Taylor's hypothesis).

Peaks in the spectrum show which size eddies contribute the most to the turbulence kinetic energy. The leftmost peak with a period of near 100 h corresponds to wind speed variations associated with the passage of fronts and weather systems. In other words, there is evidence of the Rossby-wave cycle in our wind speed record. The next peak, at 24 h, shows the diurnal increase of wind speed during the day and decrease at night. The rightmost peak is the one we will study in this book. It indicates the microscale eddies having durations of 10 s to 10 min, just what we would have guessed from examining Fig 2.1 by eye.

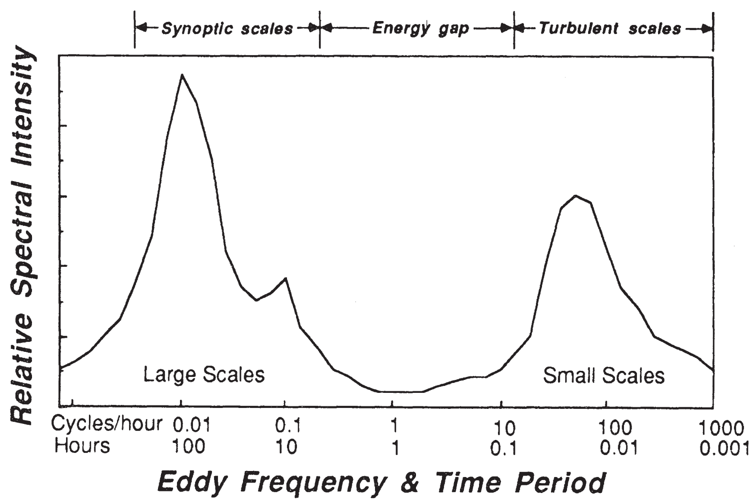


Fig. 2.2 Schematic spectrum of wind speed near the ground estimated from a study of Van der Hoven (1957).

Within the rightmost peak we see that the largest eddies are usually the most intense. The smaller, high frequency, eddies are very weak, as previously discussed. Large-eddy motions can create eddy-size wind-shear regions, which can generate smaller eddies. Such a net transfer of turbulence energy from the larger to the smaller eddies is known as the energy cascade. At the smallest size eddies, this cascade of energy is dissipated into heat by molecular viscosity. The flavor of this energy cascade was captured by Lewis Richardson in his 1922 poem:

Big whorls have little whorls,
Which feed on their velocity;
And little whorls have lesser whorls,
And so on to viscosity
(in the molecular sense).

In a later chapter we will describe the mathematical tools necessary to calculate spectra.

2.2 The Spectral Gap

There appears to be a distinct lack of wind-speed variation in Fig 2.1 having time periods of about 30 min to 1 h. The slow variation of the mean wind speed from 6 to 5 m/s over the 2 h period was already discussed. Shorter time periods were what we associated with microscale turbulence. The lack of variation at the intermediate time or space scales has been called the ***spectral gap***.

A separation of scales is evident in Fig 2.2, where the spectral gap appears as the large valley separating the microscale from the synoptic scale peaks. Motions to the left of the gap are said to be associated with the ***mean flow***. Motions to the right constitute ***turbulence***. The center of the gap is near the one-hour time period.

It is no accident that the response time used in Chapter 1 to define the BL is one hour. Implicit in the definition of the BL is the concept that turbulence is the primary agent for effecting changes in the BL. Hence, the spectral gap provides a means to separate the turbulent from the nonturbulent influences on the BL (in the microscale sense).

For some flows there might not be a spectral gap. For example, larger cumulus clouds act like large eddies with time scales on the order of an hour. Consequently, a spectrum of wind speed made in the cloud layer might not exhibit a vivid separation of scales. Most analyses of turbulence rely on the separation of scales to simplify the problem; hence, cloud-filled flow regimes might be difficult to properly describe.

Many of the operational numerical weather prediction models use grid spacings or wavelength cutoffs that fall within the spectral gap. This means that larger-scale motions can be explicitly resolved and deterministically forecast. The smaller-scale motions, namely turbulence, are not modeled directly. Rather, the effects of those subgrid scales on the larger scales are approximated. These smaller-size motions are said to be parameterized by subgrid-scale stochastic (statistical) approximations or models.

2.3 Mean and Turbulent Parts

There is a very easy way to isolate the large-scale variations from the turbulent ones. By averaging our wind speed measurements over a period of 30 minutes to one hour, we can eliminate or "average out" the positive and negative deviations of the turbulent velocities about the mean. Once we have the mean velocity, $\overline{U}$, for any time period, we can subtract it from the actual instantaneous velocity, U, to give us just the turbulent part, u':

$$u' = U - \overline{U} \tag{2.3a}$$

The existence of a spectral gap allows us to partition the flow field in this manner.

We can think of u' as the gust that is superimposed on the mean wind. It represents the part of the flow that varies with periods shorter than about one hour. The mean, $\overline{U}$, represents the part that varies with a period longer than about one hour.

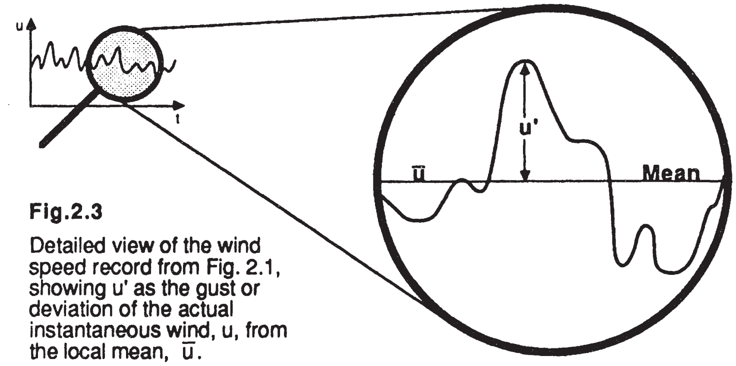


Fig.2.3

Detailed view of the wind speed record from Fig. 2.1, showing u' as the gust or deviation of the actual instantaneous wind, u, from the local mean, $\overline{u}$.

Fig 2.3 shows an expanded view of just a small portion of the wind trace from Fig 2.1. The straight line represents the mean wind over that portion of the record, while the wiggly line represents the actual instantaneous wind speed. The gust part, u', is sketched as the distance between those two lines. At some times the gust is positive, meaning the actual wind is faster than average. At other times the gust is negative, indicative of a slower than average wind.

Microscale turbulence is a three-dimensional phenomenon. Therefore, we expect that gusts in the x-direction might be accompanied by gusts in the y- and z-directions. Turbulence, by definition, is a type of motion. Yet motions frequently cause variations in the temperature, moisture, and pollutant fields if there is some mean gradient of that variable across the turbulent domain. Hence, we can partition each of the variables into mean and turbulent parts:

$$\begin{aligned} U &= \overline{U} + u' \\ V &= \overline{V} + v' \\ W &= \overline{W} + w' \\ \theta_v &= \overline{\theta_v} + \theta_v' \\ q &= \overline{q} + q' \\ c &= \overline{c} + c' \end{aligned} \tag{2.3b}$$

Each of these terms varies in time and space (see Appendix B for a list of symbols).

2.4 Some Basic Statistical Methods

Because one of the primary avenues for studying turbulent flow is the stochastic approach, it is desirable to have a good working knowledge of statistics. This section will survey some of the basic methods of statistics, including the mean, variance, standard deviation, covariance, and correlation. Those readers having an adequate background on statistics might wish to skim this section.

2.4.1 The Mean

Time $\overline{(\ )}^{t}$, space $\overline{(\ )}^{s}$, and ensemble $\overline{(\ )}^{e}$ averages are three ways to define a mean. The ***time average*** applies at one specific point in space, and consists of a sum or integral over time period P. For any variable, A(t, s), that is a function of time, t, and space, s:

$$\overline{A}^{t}(s) = \frac{1}{N}\sum_{i=0}^{N-1} A(i, s) \qquad \text{or} \qquad \overline{A}^{t}(s) = \frac{1}{P}\int_{t=0}^{P} A(t, s)\, dt \tag{2.4.1a}$$

where $t = i\,\Delta t$ for the discrete case.

The ***spatial average***, which applies at some instant in time, is given by a sum or integral over spatial domain S:

$$\overline{A}^{s}(t) = \frac{1}{N}\sum_{j=0}^{N-1} A(t,j) \qquad \text{or} \qquad \overline{A}^{s}(t) = \frac{1}{S}\int_{t=0}^{S} A(t,s)\, ds \tag{2.4.1b}$$

where $s = j\Delta s$ in the discrete case.

An ***ensemble average*** consists of the sum over N identical experiments:

$$\overline{A}^{e}(t,s) = \frac{1}{N}\sum_{i=0}^{N-1} A_i(t,s) \tag{2.4.1c}$$

In the equations above, $\Delta t = P/N$ and $\Delta s = S/N$, where N is the number of data points.

For laboratory experiments, the ensemble average is the most desirable, because it allows us to reduce random experimental errors by repeating the basic experiment. Unlike laboratory experiments, however, we have little control over the atmosphere, so we are rarely able to observe reproducible weather events. We are therefore unable to use the ensemble average.

Spatial averages are possible by deploying an array of meteorological sensors covering a line, area, or volume. If the turbulence is ***homogeneous*** (statistically the same at every point in space, see Fig 2.4) then each of the sensors in the array will be measuring the same phenomenon, making a spatial average meaningful. The real atmosphere, however, is horizontally homogeneous in only limited locations, meaning that most spatial means are averaged over a variety of different phenomena. By proper choice of sensor-array domain size as well as intra-array spacing, one can sometimes isolate scales of phenomena for study, while averaging out the other scales.

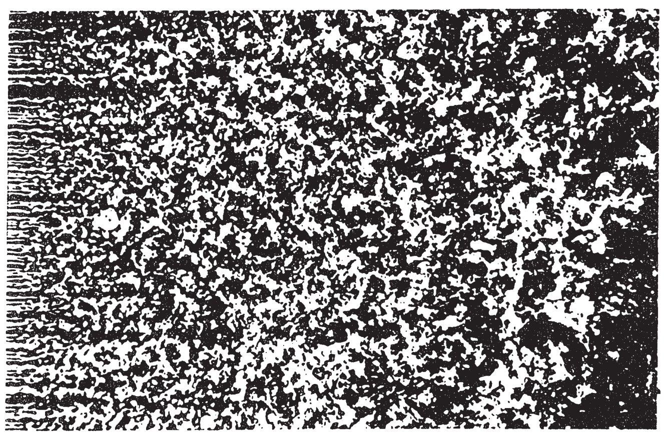


Fig. 2.4 Laboratory generation of homogeneous turbulence behind a grid. Using a finer grid than Fig 2.4, the merging unstable wakes quickly form a homogeneous field. As it decays downstream, it provides a useful approximation of homogeneous turbulence.(From Van Dyke, 1982)

Volume averaging is virtually impossible using direct sensors such as thermometers because of the difficulty of deploying these sensors at all locations and altitudes throughout the BL. Remote sensors such as radars, lidars, and sodars, however, can scan volumes of the atmosphere, making volume averages of selected variables possible. Details of these sensors are discussed in chapter 10.

Area averaging in the surface layer is frequently performed within small domains by deploying an array of small instrumented masts or instrument shelters on the ground. ***Line averages*** are similarly performed by erecting sensors along a road, for example.

Sensors mounted on a moving platform, such as a truck or an aircraft, can provide quasi-line averages. These are not true line (spatial) averages because the turbulence state of the flow may change during the time it takes the platform to move along the desired path. As a result, most measurement paths are designed as a compromise between long length (to increase the statistical significance by observing a larger number of data points) and short time (because of the diurnal changes that occur in the mean and turbulent state over most land surfaces).

Time averages are frequently used, and are computed from sensors mounted on a single, fixed-location platform such as a mast or tower. The relative ease of making observations at a fixed point has meant that time averaging has been the most popular in the lower BL. Some vertically-looking remote sensors also use this method to observe the middle and top of the BL. For turbulence that is both homogeneous and ***stationary*** (statistically not changing over time), the time, space and ensemble averages should all be equal. This is called the ***ergodic*** condition, which is often assumed to make the turbulence problem more tractable:

$$ {}^{e}\overline{(\ )} = {}^{t}\overline{(\ )} = {}^{s}\overline{(\ )} \equiv \overline{(\ )} \tag{2.4.1d} $$

This book will use the overbar $\overline{(\ )}$ as an abbreviation for a generic average, not specifying whether it is a time, space, or ensemble average. Because of the popularity of the time average, however, many of our examples will use time as the independent variable.

2.4.2 Rules of Averaging

Let A and B be two variables that are dependent on time, and let c represent a constant. To find the average of the sum of A and B, we can employ the equations of the previous section with some basic rules of summation or integration to show that:

$$ \overline{(A+B)} = \overline{A} + \overline{B} \tag{2.4.2a} $$

In terms of discrete sums, the average is:

$$ \begin{aligned} \overline{(A+B)} &= \frac{1}{N}\sum_{i=0}^{N-1}(A_i + B_i) \\ &= \frac{1}{N}\left(\sum_i A_i + \sum_i B_i\right) \\ &= \frac{1}{N}\sum_i A_i + \frac{1}{N}\sum_i B_i \\ &= \overline{A} + \overline{B} \end{aligned} $$

In terms of continuous integrals:

$$\overline{(A+B)} = \frac{1}{P}\int_{t=0}^{P}(A+B)\,dt$$

$$= \frac{1}{P}\left(\int_t A\,dt + \int_t B\,dt\right)$$

$$= \frac{1}{P}\int_t A\,dt + \frac{1}{P}\int_t B\,dt$$

$$= \overline{A} + \overline{B}$$

Both the sum and integral approaches give the same answer, as expected.

We can use similar methods to show that:

$$\overline{(c\,A)} = c\,\overline{(A)} \qquad (2.4.2b)$$

$$\overline{c} = c \qquad (2.4.2c)$$

An important consequence of averaging is that an average value acts like a constant when averaged a second time over the same time period, P:

Define

$$\frac{1}{P}\int_{t=0}^{P} A(t,s)\,dt \equiv \overline{A}(P,s)$$

Therefore

$$\frac{1}{P}\int_{t=0}^{P}\overline{A}(P,s)\,dt \equiv \overline{A}(P,s)\,\frac{1}{P}\int_{t=0}^{P} dt$$

$$= \overline{A}(P,s)$$

Leaving

$$\overline{\overline{(A)}} = \overline{A} \qquad (2.4.2d)$$

Similarly, it can be shown that:

$$\overline{(\overline{A}\ B)} = \overline{A}\ \overline{B} \tag{2.4.2e}$$

Often we need to find the average of a derivative of a dependent variable. For example, let A be dependent on both t and s, where s is an independent variable such as x, y, or z. In this case, we must use ***Leibniz' theorem***:

$$\frac{d}{dt}\left[\int_{S_1(t)}^{S_2(t)} A(t,s)\, ds\right] = \int_{S_1(t)}^{S_2(t)} \left[\frac{\partial A(t,s)}{\partial t}\right] ds + A(t, S_2)\frac{d S_2}{dt} - A(t, S_1)\frac{d S_1}{dt} \tag{2.4.2f}$$

where S_1 and S_2 are the limits of integration.

For the special case where S_1 and S_2 are constant with time, we can simply interchange the order of integration and differentiation:

$$\frac{d}{dt}\left[\int_s A\, ds\right] = \int_s \left[\frac{\partial A}{\partial t}\right] ds$$

Multiplying both sides by 1/S gives, where $S = S_2 - S_1$:

$$\frac{d\left({}^{s}\overline{A}\right)}{dt} = {}^{s}\overline{\left(\frac{\partial A}{\partial t}\right)} \tag{2.4.2g}$$

This special case is not always valid for variable depth boundary layers.

Suppose we wish to find the time-rate-of-change of a BL-averaged mixing ratio, $\bar{r}$, where the BL average is defined by integrating over the depth of the BL; i.e., from z=0 to $z=z_i$. Since z_i varies with time, we can use the full Leibniz' theorem to give:

$$\frac{d}{dt}\left[z_i\ {}^{s}\bar{r}\right] = z_i\ {}^{s}\overline{\left[\frac{\partial r}{\partial t}\right]} + r(t, z_i^+)\frac{d z_i}{dt} \tag{2.4.2h}$$

where z_i^+ represents a location just above the top of the BL.

Finally, let's re-examine the spectral gap. If our averaging time is 30 minutes to 1 hour, turbulent fluctuations will be eliminated, leaving the longer-period time variations. As we saw in Fig 2.1, the 30-minute mean wind speed changes over the period of a few hours. Thus, we can take the 30-minute average of the time-derivative of variable A to find how $\overline{A}$ varies over longer periods:

$$\overline{\left(\frac{d\,A}{dt}\right)}^{t} = \frac{d\,\overline{A}^{t}}{dt} \qquad (2.4.2i)$$

In other words, the average of the local slopes (slope = rate of change with time) equals the slope of the averages (see Fig 2.5).

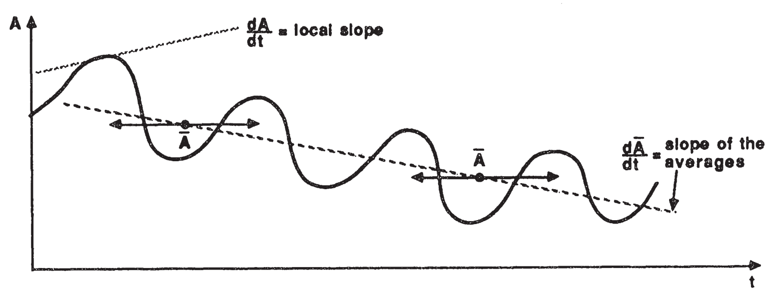


Fig. 2.5 Schematic comparison of the slope of average values verses the average of local slopes. Averaging period, P, is indicated by the width of the horizontal arrows.

This is a difficult concept that deserves some thought on the part of the reader. It is an important consequence of the spectral gap because it allows us to make a deterministic forecast of a mean variable such as $\overline{A}$ using simplified, stochastic, representations of the turbulence. Otherwise, operational forecasts of seemingly simple variables such as temperature or wind would be much more difficult.

To summarize the rules of averaging:

$$\begin{aligned}
\overline{c} &= c \\
\overline{(c\,A)} &= c\,\overline{A} \\
\overline{(\overline{A})} &= \overline{A} \\
\overline{(\overline{A}\,B)} &= \overline{A}\,\overline{B} \\
\overline{(A+B)} &= \overline{A} + \overline{B} \\
\overline{\left(\frac{d\,A}{dt}\right)} &= \frac{d\,\overline{A}}{dt}
\end{aligned} \qquad (2.4.2k)$$

2.4.3 Reynolds Averaging

The averaging rules of the last section can now be applied to variables that are split into mean and turbulent parts. Let $A = \overline{A} + a'$ and $B = \overline{B} + b'$. Starting with the instantaneous value, A, for example, we can find its mean using the fifth and third rules of the previous section:

$$\overline{(A)} = \overline{(\overline{A} + a')} = \overline{(\overline{A})} + \overline{a'} = \overline{A} + \overline{a'}$$

The only way that the left and right sides can be equal is if

$$\overline{a'} = 0 \qquad (2.4.3a)$$

This result is not surprising if one remembers the definition of a mean value. By definition, the sum of the positive deviations from the mean must equal the sum of the negative deviations. Thus the deviations balance when summed, as implied in the above average.

Another example: start with the product $\overline{B}\, a'$ and find its average. Employing the above result together with the fourth averaging rule, we find that

$$\overline{(\overline{B}\, a')} = \overline{B}\,\overline{a'} = \overline{B} \cdot 0 = 0 \qquad (2.4.3b)$$

Similarly, $\overline{\overline{A}\, b'} = 0$. One should not become too lax about the average of primed variables, as is demonstrated next.

The average of the product of A and B is

$$\begin{aligned} \overline{(A \cdot B)} &= \overline{(\overline{A} + a')\,(\overline{B} + b')} \\ &= \overline{(\overline{A}\,\overline{B} + a'\,\overline{B} + \overline{A}\,b' + a'\,b')} \\ &= \overline{(\overline{A}\,\overline{B})} + \overline{(a'\,\overline{B})} + \overline{(\overline{A}\,b')} + \overline{(a'\,b')} \\ &= \overline{A}\,\overline{B} + 0 + 0 + \overline{a'\,b'} \\ &= \overline{A}\,\overline{B} + \overline{a'\,b'} \end{aligned} \qquad (2.4.3c)$$

The nonlinear product $\overline{a'\,b'}$ is NOT necessarily zero. The same conclusion holds for other nonlinear variables such as:

$$\overline{a'^2} \ , \ \overline{a'b'^2} \ , \ \overline{a'^2 b'^2} \ .$$

In fact, these nonlinear terms must be retained to properly model turbulence. This is a dramatic difference from many linear theories of waves, where the nonlinear terms are often neglected as a first-order approximation.

2.4.4 Variance, Standard Deviation and Turbulence Intensity

One statistical measure of the dispersion of data about the mean is the variance, σ^2, defined by:

$$\sigma_A^2 = \frac{1}{N} \sum_{i=0}^{N-1} (A_i - \overline{A})^2 \qquad (2.4.4a)$$

This is known as the ***biased variance***. It is a good measure of the dispersion of a sample of BL observations, but not the best measure of the dispersion of the whole population of possible observations. A better estimate of the variance (an ***unbiased variance***) of the population, given a sample of data, is

$$\sigma_A^2 = \frac{1}{(N-1)} \sum_{i=0}^{N-1} (A_i - \overline{A})^2 \qquad (2.4.4b)$$

When N is large, as it often is for turbulence measurements, $1/N \cong 1/(N-1)$. As a result, the biased definition is usually used in BL meteorology for convenience.

Recall that the turbulent part (or the perturbation or gust part) of a turbulent variable is given by $a' = A - \overline{A}$. Substituting this into the biased definition of variance gives

$$\sigma_A^2 = \frac{1}{N} \sum_{i=0}^{N-1} a'^{\,2}_i = \overline{a'^2} \qquad (2.4.4c)$$

Thus, whenever we encounter the average of the square of a turbulent part of a variable, such as $\overline{u'^2}$, $\overline{v'^2}$, $\overline{w'^2}$, $\overline{\theta'^2}$, $\overline{r'^2}$, or $\overline{q'^2}$, we can interpret these as variances.

The standard deviation is defined as the square root of the variance:

$$\sigma_A = \left(\overline{a'^2}\right)^{1/2} \qquad (2.4.4d)$$

The standard deviation always has the same dimensions as the original variable. Fig 2.6 shows the relationship between a turbulent trace of wind speed and the corresponding standard deviation. It can be interpreted as a measure of the magnitude of the spread or dispersion of the original data from its mean. For this reason, it is used as a measure of the intensity of turbulence. In figure 2.1, for example, we might guess the standard deviation to be about 0.5 - 0.6 m/s at noon, dropping to about 0.3 m/s by 1400 local time.

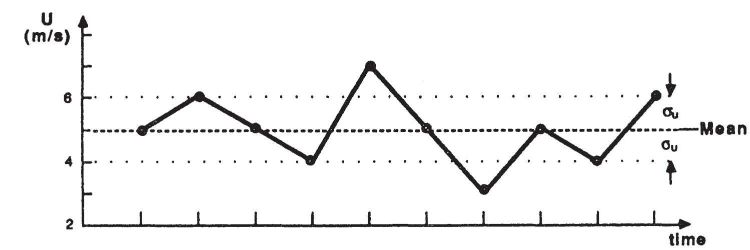


Fig. 2.6 Relationship between the standard deviation and turbulence variations. Solid line connects the data points; heavy dashed line is the average; dotted lines are drawn one standard deviation above and below the mean. Based on the example data in section 2.4.5.

Near the ground, the turbulence intensity might be expected to increase as the mean wind speed, M, increases. For this reason a dimensionless measure of the ***turbulence intensity***, I, is often defined as

$$I = \sigma_M / \overline{M} \qquad (2.4.4e)$$

For mechanically generated turbulence, one might expect σ_M to be a simple function of M. As we learned in Chapter 1, $I < 0.5$ is required for Taylor's hypothesis to be valid.

2.4.5 Covariance and Correlation

In statistics, the covariance between two variables is defined as

$$\text{covar}(A,B) \equiv \frac{1}{N} \sum_{i=0}^{N-1} (A_i - \overline{A}) \cdot (B_i - \overline{B}) \qquad (2.4.5a)$$

Using our Reynolds averaging methods, we can show that:

$$\text{covar}(A,B) \equiv \frac{1}{N} \sum_{i=0}^{N-1} a_i' \, b_i'$$

$$= \overline{a' \, b'} \qquad (2.4.5b)$$

Thus, the nonlinear turbulence products that were introduced in section 2.4.3 have the same meaning as covariances.

The covariance indicates the degree of common relationship between the two variables, A and B. For example, let A represent air temperature, T, and let B be the vertical velocity, w. On a hot summer day over land, we might expect the warmer than average air to rise (positive T' and positive w'), and the cooler than average air to sink (negative T' and negative w'). Thus, the product w'T' will be positive on the average, indicating that w and T vary together. The covariance $\overline{w'T'}$ is indeed found to be positive throughout the bottom 80% of the convective mixed layer.

Sometimes, one is interested in a normalized covariance. Such a relationship is defined as the ***linear correlation coefficient***, r_{AB}:

$$r_{AB} \equiv \frac{\overline{a' \, b'}}{\sigma_A \, \sigma_B} \qquad (2.4.5c)$$

This variable ranges between -1 and +1 by definition. Two variables that are perfectly correlated (i.e., vary together) yield r = 1. Two variables that are perfectly negatively correlated (i.e., vary oppositely) yield r = -1. Variables with no net variation together yield r = 0. Fig 2.7 shows typical correlation coefficients in the ML.

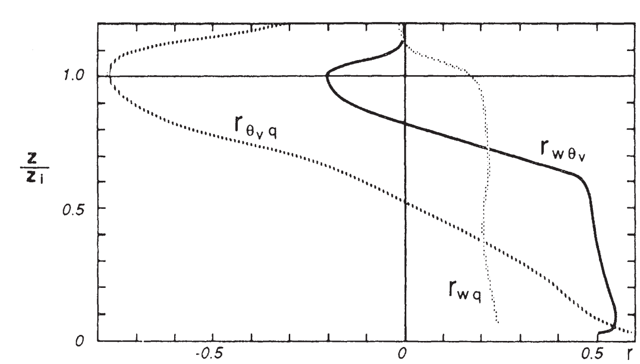


Fig. 2.7 Correlation coefficient profiles in the convective mixed layer.

2.4.6 Example

Problem. Suppose that we erect a short mast instrumented with anemometers to measure the U and W wind components. We record the instantaneous wind speeds every 6 s for a minute, resulting in the following 10 pairs of wind observations:

U (m/s):	5	6	5	4	7	5	3	5	4	6
W (m/s):	0	-1	1	0	-2	1	2	-1	1	-1

Find the mean, biased variance, and standard deviation for each wind component. Also, find the covariance and correlation coefficient between U and W.

Solutions.

$$\overline{U} = 5\ \text{m·s}^{-1} \qquad \sigma_U^2 = 1.20\ \text{m}^2\text{·s}^{-2} \qquad \sigma_U = 1.10\ \text{m·s}^{-1}$$

$$\overline{W} = 0\ \text{m·s}^{-1} \qquad \sigma_W^2 = 1.40\ \text{m}^2\text{·s}^{-2} \qquad \sigma_W = 1.18\ \text{m·s}^{-1}$$

$$\overline{u'w'} = -1.10\ \text{m}^2\text{·s}^{-2} \qquad r_{UW} = -0.85\ \text{(dimensionless)}$$

Discussion. Thus, the turbulent variations of W are more intense than those of U, even though the mean wind speed for W is zero in this example. U and W tend to vary in opposite directions on the average, as indicated by the negative values for the covariance and the correlation coefficient. The magnitude of the correlation coefficient is fairly high (close to one), meaning that there are just a few observations where U and W vary in the same direction, but many more observations where they vary oppositely.

2.5 Turbulence Kinetic Energy

The usual definition of kinetic energy (KE) is KE = $0.5\ m\ M^2$, where m is mass. When dealing with a fluid such as air it is more convenient to talk about kinetic energy per unit mass, which is just $0.5\ M^2$.

It is enticing to partition the kinetic energy of the flow into a portion associated with the mean wind (MKE), and a portion associated with the turbulence (TKE). By taking advantage of the mean and turbulent parts of velocity introduced in section 2.3, we can immediately write the desired equations:

$$\text{MKE}/m = \frac{1}{2}\left(\overline{U}^2 + \overline{V}^2 + \overline{W}^2\right) \tag{2.5a}$$

$$e = \frac{1}{2}\left(u'^2 + v'^2 + w'^2\right) \tag{2.5b}$$

where e represents an instantaneous turbulence kinetic energy per unit mass. There is an additional portion of the total KE consisting of mean-turbulence products, but this disappears upon averaging.

Rapid variations in the value of e with time can be expected as we measure faster and slower gusts. By averaging over these instantaneous values, we can define a mean turbulence kinetic energy (TKE) that is more representative of the overall flow:

$$\frac{TKE}{m} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) = \bar{e} \qquad (2.5c)$$

We can immediately see the relationship between TKE/m and the definition of variance defined in the last section. It is apparent that statistics will play an important role in our quantification of turbulence.

The turbulence kinetic energy is one of the most important quantities used to study the turbulent BL. We have already discussed in Chapter 1 that turbulence can be generated by buoyant thermals and by mechanical eddies. It is suppressed by statically stable lapse rates and dissipated into heat by the effects of molecular viscosity. By writing a budget equation for TKE, we can balance the production terms against the loss terms to determine whether the BL will become more turbulent, or whether turbulence will decay in the BL. This will be done in Chapter 5.

A typical daytime variation of TKE in convective conditions is shown in Fig 2.8. Examples of the vertical profile of TKE for various boundary layers are shown in Fig 2.9. During the daytime, buoyancy allows air parcels to accelerate in the middle of the ML, allowing $\overline{w'^2}$ to be large there and contributing to the total TKE (Fig 2.9a).

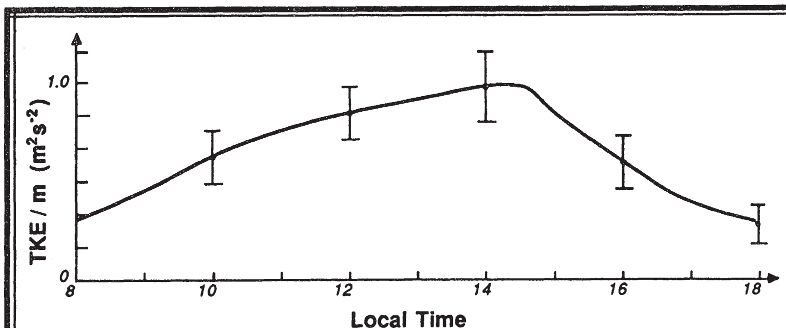


Fig. 2.8 Diurnal cycle of TKE/m = 1/2 ($\overline{u'^2} + \overline{v'^2} + \overline{w'^2}$) measured by aircraft below 300 m agl over Tennessee, August 1978. Adapted from Bean and Repoff, 1983.

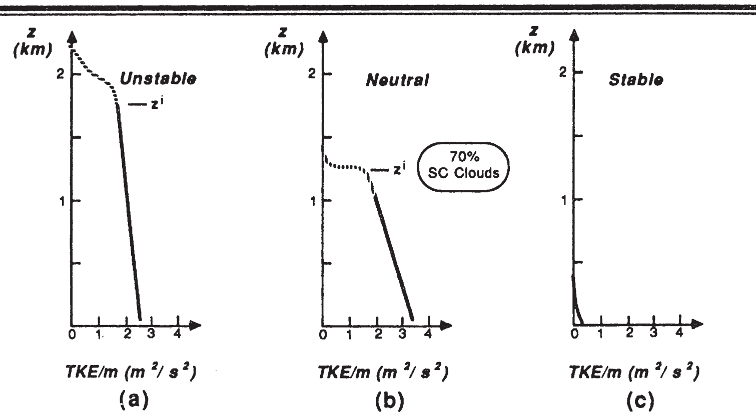


Fig. 2.9 Examples of turbulence kinetic energy per unit mass (TKE/m = 1/2 ($\overline{u'^2} + \overline{v'^2} + \overline{w'^2}$)) for various boundary layers: (a) Daytime convective mixed layer with mostly clear skies and light winds 1600 local time, 14 June 1983, Chickasha, OK; (b) Near neutral day with strong winds (10-15 m/s at surface) and broken clouds at 1100 local time, 12 June 1983, Chickasha, OK; (c) Nocturnal stable boundary layer at 1000 local time, 11 September 1973 in Minnesota (from Caughey, et al, 1979,) Part (c) data excludes $\overline{v'^2}$ contributions because of measurement errors.

On overcast days when there is little heating of the ground, wind shears and flow over obstacles create turbulence near the ground that gradually decreases intensity with height (Fig 2.9b). This turbulence is produced in primarily the $\overline{u'^2}$ and $\overline{v'^2}$ components. Days of both strong winds and strong heating will have both sources of turbulence.

For night, Fig 2.9c shows how the static stability suppresses the TKE, causing it to decrease rapidly with height. Turbulence is produced primarily near the ground by wind shears, although the enhanced shears near the nocturnal jet can also generate turbulence. Not apparent in this figure is the observation that nocturnal turbulence is sometimes sporadic: happening in turbulent bursts followed by quiescent periods.

2.6 Kinematic Flux

2.6.1 Definitions

Flux is the transfer of a quantity per unit area per unit time. In BL meteorology, we are often concerned with mass, heat, moisture, momentum and pollutant fluxes. The dimensions of these fluxes are summarized below, using SI units as the example:

Flux	Symbol	Units
mass	$\tilde{M}$	$\left[\frac{kg_{air}}{m^2 \cdot s}\right]$
heat	$\tilde{Q}_H$	$\left[\frac{J}{m^2 \cdot s}\right]$
moisture	$\tilde{R}$	$\left[\frac{kg_{water}}{m^2 \cdot s}\right]$
momentum	$\tilde{F}$	$\left[\frac{kg \cdot (m \cdot s^{-1})}{m^2 \cdot s}\right]$
pollutant	$\tilde{\chi}$	$\left[\frac{kg_{pollutant}}{m^2 \cdot s}\right]$ or $\left[\frac{kg_{pollutant}}{m^3} \cdot \frac{m}{s}\right]$

Sometimes the moisture flux is rewritten as a latent heat flux, $\tilde{Q}_E$, where $\tilde{Q}_E = L_v \tilde{R}$ and L_v is the latent heat of vaporization of water ($L_v \cong 2.45 \times 10^6$ J/kg at a summertime BL temperature of 20 °C).

As a reminder, momentum is mass times velocity (kg·m/s); thus, a momentum flux is (kg·m/s)/(m^2·s). These units are identical to N/m^2, which are the units for stress. The nature of stress is reviewed in section 2.9.

Unfortunately, we rarely measure quantities such as heat or momentum directly. Instead we measure things like temperature or wind speed. Therefore, for convenience the above fluxes can be redefined in ***kinematic form*** by dividing by the density of moist air, ρ_{air}. In the case of sensible heat flux, we also divide by the specific heat of air. In fact, the term $\rho\, C_p = 1.216 \times 10^3$ (W/m^2) / (K·m/s) allows us to easily convert between kinematic heat fluxes and normal heat fluxes.

Kinematic Flux	Symbol	Equation	Units	
mass		$M = \frac{\tilde{M}}{\rho_{air}}$	$\left[\frac{m}{s}\right]$	(2.6.1a)
heat		$Q_H = \frac{\tilde{Q}_H}{\rho_{air}\, C_{p_{air}}}$	$\left[K \frac{m}{s}\right]$	(2.6.1b)

$$\text{moisture} \qquad R = \frac{\tilde{R}}{\rho_{air}} \qquad \left[\frac{kg_{water}}{kg_{air}} \cdot \frac{m}{s} \right] \qquad (2.6.1c)$$

$$\text{momentum} \qquad F = \frac{\tilde{F}}{\rho_{air}} \qquad \left[\frac{m}{s} \cdot \frac{m}{s} \right] \qquad (2.6.1d)$$

$$\text{pollutant} \qquad \chi = \frac{\tilde{\chi}}{\rho_{air}} \qquad \left[\frac{kg_{pollutant}}{kg_{air}} \cdot \frac{m}{s} \right] \qquad (2.6.1e)$$

The above definitions are viable because the boundary layer is usually so thin that the density change across it can be neglected in comparison to changes of the other meteorological variables. For example, the standard atmospheric air density is 1.225 kg/m^3 at sea level and 1.112 kg/m^3 at 1000m, a difference of only 10%.

These kinematic fluxes are now expressed in units that we can measure directly: wind speed for mass and momentum fluxes; temperature and wind speed for heat flux; and specific humidity (q) and wind speed for moisture flux. The pollutant flux is frequently expressed in either form: concentration and wind speed, or mass ratio (like parts per million, ppm) and wind speed.

Each of these fluxes can be split into three components. For example, there might be a vertical component of heat flux, and two horizontal components of heat flux, as sketched in Fig 2.10. Similar fluxes could be expected for mass, moisture, and pollutants. Hence, we can picture these fluxes as vectors.

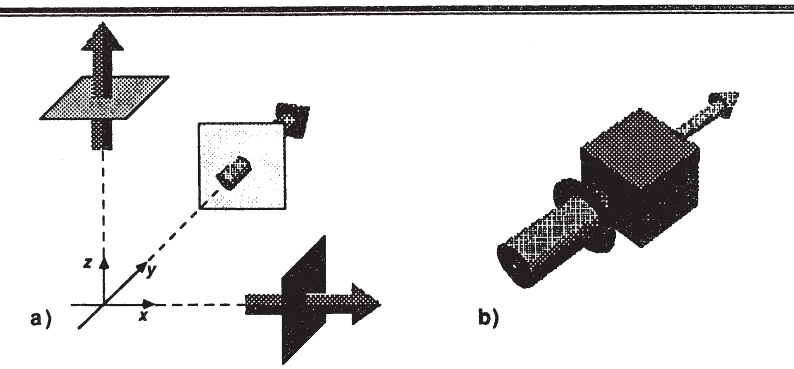
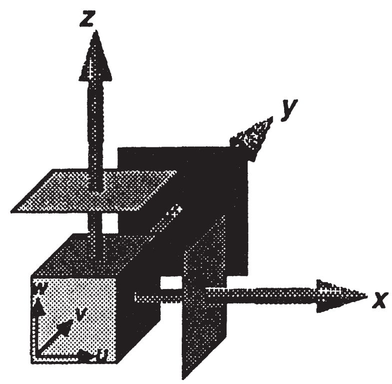


Fig. 2.10 Flux is the rate of transfer of a quantity across a unit area. (a) The net flux can be split into components in the x,y, and z directions, as shown here. (b) If a greater flux enters a volume than leaves, then there must be a net increase of that quantity within the volume.

For momentum, we have the added dimension that the flux in any one direction might be the flux of U, V or W momentum (see Fig 2.11). This means that there are nine components of this flux to consider: each of the three momentum components can pass through a plane normal to any of the three cartesian directions. The momentum flux is thus said to be a ***tensor***. Just for the record, this kind of tensor is known as a second order tensor. A vector is a first order tensor, and a scalar is a zero order tensor.

Fig. 2.11
Momentum can be split into the three cartesion directions, based on the u, v, and w components of wind. Momentum flux can consist of the transfer of any of these three components in any of three directions: x, y, and z, yielding a total of nine momentum flux components.



As you might guess, we can split the fluxes into mean and turbulent parts. For the flux associated with the mean wind (i.e., advection), it is easy to show, for example, that

Vertical kinematic advective heat flux	=	$\overline{W}\cdot\overline{\theta}$	(2.6.1f)
Vertical kinematic advective moisture flux	=	$\overline{W}\cdot\overline{q}$	(2.6.1g)
x-direction kinematic advective heat flux	=	$\overline{U}\cdot\overline{\theta}$	(2.6.1h)
Vertical kinematic advective flux of U-momentum	=	$\overline{W}\cdot\overline{U}$	(2.6.1i)

The last flux is also the kinematic flux of W-momentum in the x-direction.

Fluxes in other directions can be constructed in an analogous fashion. These fluxes have the proper dimensions for kinematic fluxes. They also make physical sense. For example, a greater vertical velocity or a greater potential temperature both create a greater vertical heat flux, as would be intuitively expected.

2.6.2 Example

Problem. Given $\tilde{Q}_H = 365\ \mathrm{W{\cdot}m^{-2}}$. Find Q_H.

Solution. $Q_H = \tilde{Q}_H / (\rho\ C_p)$

$= (365\ \mathrm{Wm^{-2}}) / [(1.21\ \mathrm{kg{\cdot}m^{-3}})\cdot(1005\ \mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}})] = 0.30\ \mathrm{K{\cdot}m/s}$

Discussion. This is a typical daytime kinematic heat flux during strong convection.

2.7 Eddy Flux

We saw in the last section that fluid motion can transport quantities, resulting in fluxes. Turbulence also involves motion. Thus we expect that turbulence transports quantities too.

2.7.1 Concepts

A term like $\overline{w'\theta'}$ looks similar to the kinematic flux terms of the last section, except that the perturbation values are used instead of the mean values of W and θ. If turbulence is completely random, then a positive $w'\theta'$ one instant might cancel a negative $w'\theta'$ at some later instant, resulting in a near zero value for the average turbulent heat flux. As is shown below, however, there are situations where the average turbulent flux might be significantly different from zero.

As a conceptual tool, suppose we examine a small idealized eddy near the ground on a hot summer day (see Fig 2.12a). The average potential temperature profile is usually superadiabatic in such surface layers. If the eddy is a swirling motion, then some of the air from position 1 will be mixed downward (i.e., w' is negative) , while some air from position 2 will mix up (i.e., w' is positive) to take its place. The average motion caused by turbulence is $\overline{w'} = 0$, as expected (from section 2.4.3).

The downward moving air parcel (negative w') ends up being cooler than its surroundings (negative θ', assuming that θ' was conserved during its travel), resulting in an instantaneous product $w'\theta'$ that is positive. The upward moving air (positive w') is warmer than its surroundings (positive θ'), also resulting in a positive instantaneous product $w'\theta'$. Both the upward and downward moving air contribute positively to the flux, $w'\theta'$; thus, the average kinematic eddy heat flux $\overline{w'\theta'}$ is positive for this small-eddy mixing process.

This important result shows that turbulence can cause a net transport of a quantity such as heat ($\overline{w'\theta'} \neq 0$), even though there is no net transport of mass ($\overline{w'} = 0$). Turbulent eddies transport heat upward in this case, tending to make the lapse rate more adiabatic.

Next, let's examine what happens on a night where a statically stable lapse rate is present (Fig 2.12b). Again, picture a small eddy moving some air up and some back down. An upward moving parcel ends up cooler than its surrounding (negative $w'\theta'$), while a downward moving parcel is warmer (negative $w'\theta'$). The net effect of the small eddy is to cause a negative $\overline{w'\theta'}$, meaning a downward transport of heat.

Fruit growers utilize this process on cold nights as one method to prevent their fruit from freezing. They run motor-driven fans throughout the orchard to generate turbulent eddies. These eddies mix the warmer air down towards the fruit, and mix the cooler near-surface air upward out of the orchard, thereby potentially saving the crop.

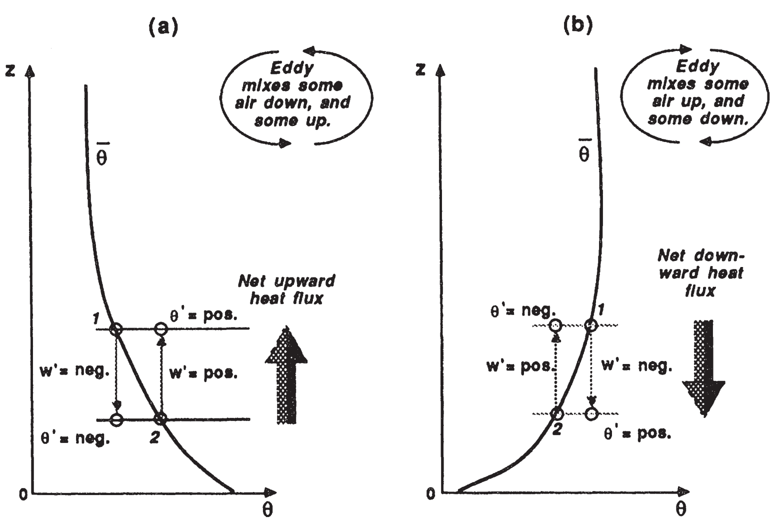


Fig. 2.12 Idealization of the small eddy mixing process, showing (a) net upward turbulent heat flux in a statically unstable environment, and (b) net downward turbulent heat flux in a stable environment.

Again we see the statistical nature of our description of turbulence. A kinematic flux such as $\overline{w'\theta'}$ is nothing more than a statistical covariance. We will usually leave out the word "kinematic" in future references to such fluxes.

As before, we can extend our arguments to write various kinds of eddy flux:

$$\text{Vertical kinematic eddy heat flux} = \overline{w'\theta'} \qquad (2.7.1a)$$

$$\text{Vertical kinematic eddy moisture flux} = \overline{w'q'} \qquad (2.7.1b)$$

$$\text{x-direction kinematic eddy heat flux} = \overline{u'\theta'} \qquad (2.7.1c)$$

$$\text{Vertical kinematic eddy flux of U-momentum} = \overline{u'w'} \qquad (2.7.1d)$$

The last flux is also the x-direction kinematic eddy flux of W-momentum.

Comparing the advective fluxes to the eddy fluxes, it is important to recognize that $\overline{W} \cong 0$ throughout most of the boundary layer. As a result, the vertical advective fluxes are usually negligible compared to the vertical turbulent fluxes. No such statement can be made about the horizontal fluxes, where strong mean horizontal winds and strong turbulence can cause fluxes of comparable magnitudes.

Finally, it is important to note that turbulence in the real atmosphere usually consists of many large positive and negative values of the instantaneous fluxes, such as heat flux $w'\theta'$. Only after averaging does a smaller, but significant, net flux $\overline{w'\theta'}$ become apparent.

An example of the instantaneous heat flux values measured by an aircraft flying near the ground during the 1983 Boundary Layer Experiment (BLX83, see Stull & Eloranta, 1984) is shown in Fig. 2.13. The net flux for this case was $\overline{w'\theta'} = 0.062 \text{ K·m·s}^{-1}$. We see from the figure that most of the time there are small positive and negative values of $w'\theta'$ that average to near zero. Occasional large positive spikes associated with convective plumes cause the net average value of $\overline{w'\theta'}$ to be positive for this case. Figs 2.14 show the corresponding histograms of frequency of occurrence of w', θ', and $w'\theta'$. The aircraft was flying at about 75 m/s, so it is easy to convert the time axis of Fig 2.13 to a distance axis. Statistics for this afternoon case are shown in table 2-1.

Table 2-1. Statistics for the 100 s segment time series shown in Fig 2.14. This segment is extracted from a 4 min flight leg near the surface, during the BLX83 field experiment. The 4 min mean values were used as the reference from which the perturbation values were calculated, which explains why $\overline{w'}$ and $\overline{\theta'}$ are not exactly zero for this 100 s segment.

Statistic	w' (m/s)	θ' (K)	w'θ' (K m/s)
Average	0.017	-0.017	0.062
σ	0.67	0.18	0.14
Maximum	2.51	0.87	1.93
Minimum	-2.07	-0.44	-0.38

2.7.2 Turbulent Flux Profiles

Fig 2.15 shows idealizations of the turbulent heat, momentum, and moisture fluxes for both the daytime and nighttime BLs. During the daytime, the fluxes are large, and usually change linearly with height over the ML. At night, the fluxes are much weaker.

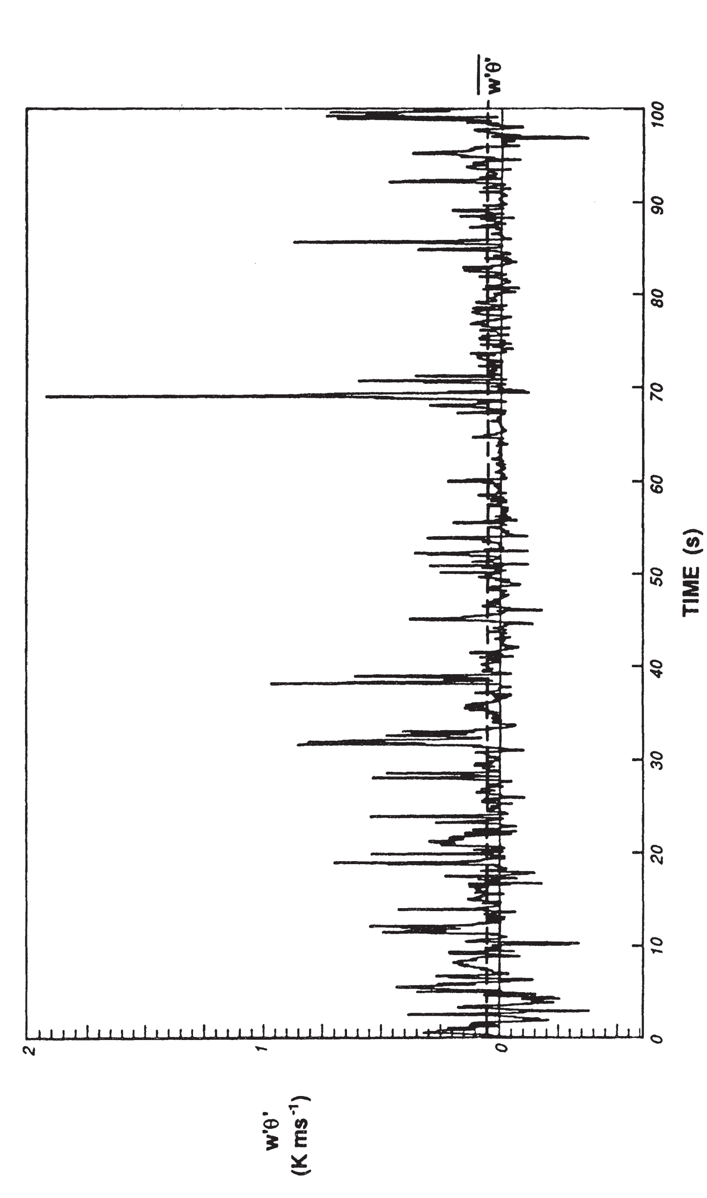


Fig. 2.13 Instantaneous kinematic surface heat flux. w'θ' trace measured by aircraft (BLX83 flight 3, leg 3) May 28, 1983 Oklahoma. The dashed line indicates the average heat flux.

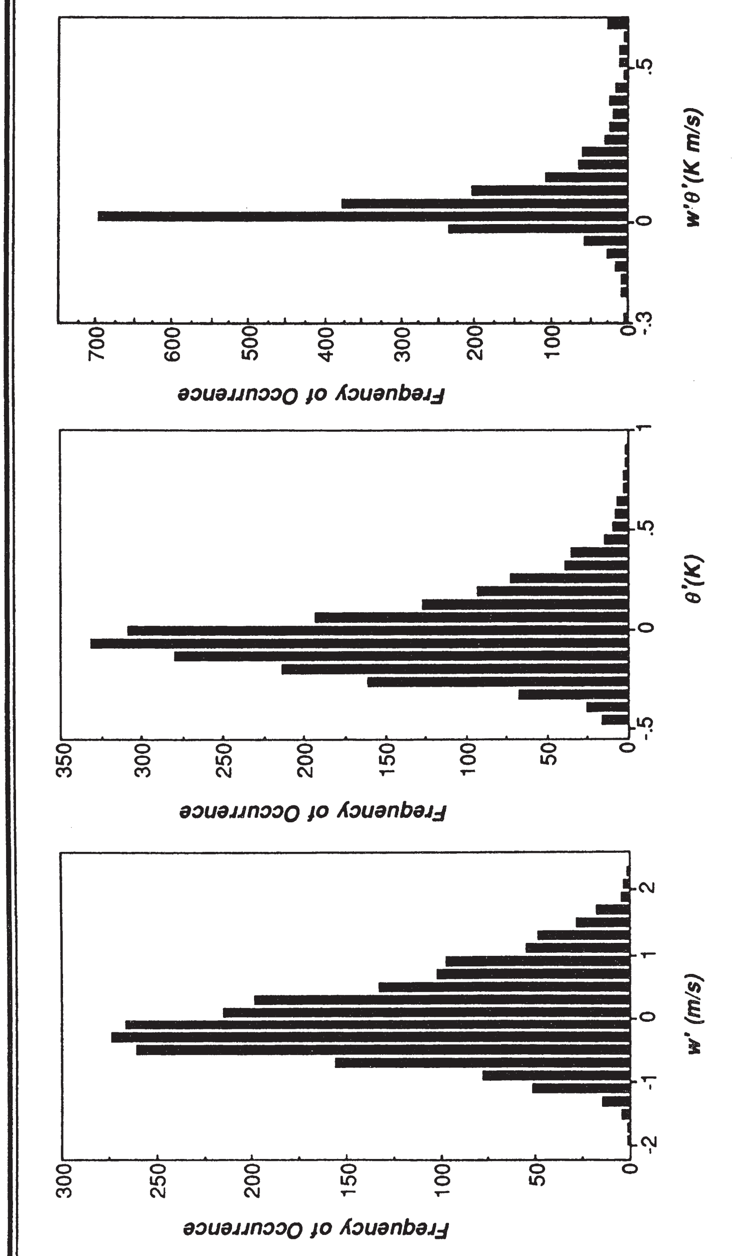


Fig. 2.14 Frequency of occurrence of w', θ', and w'θ'.

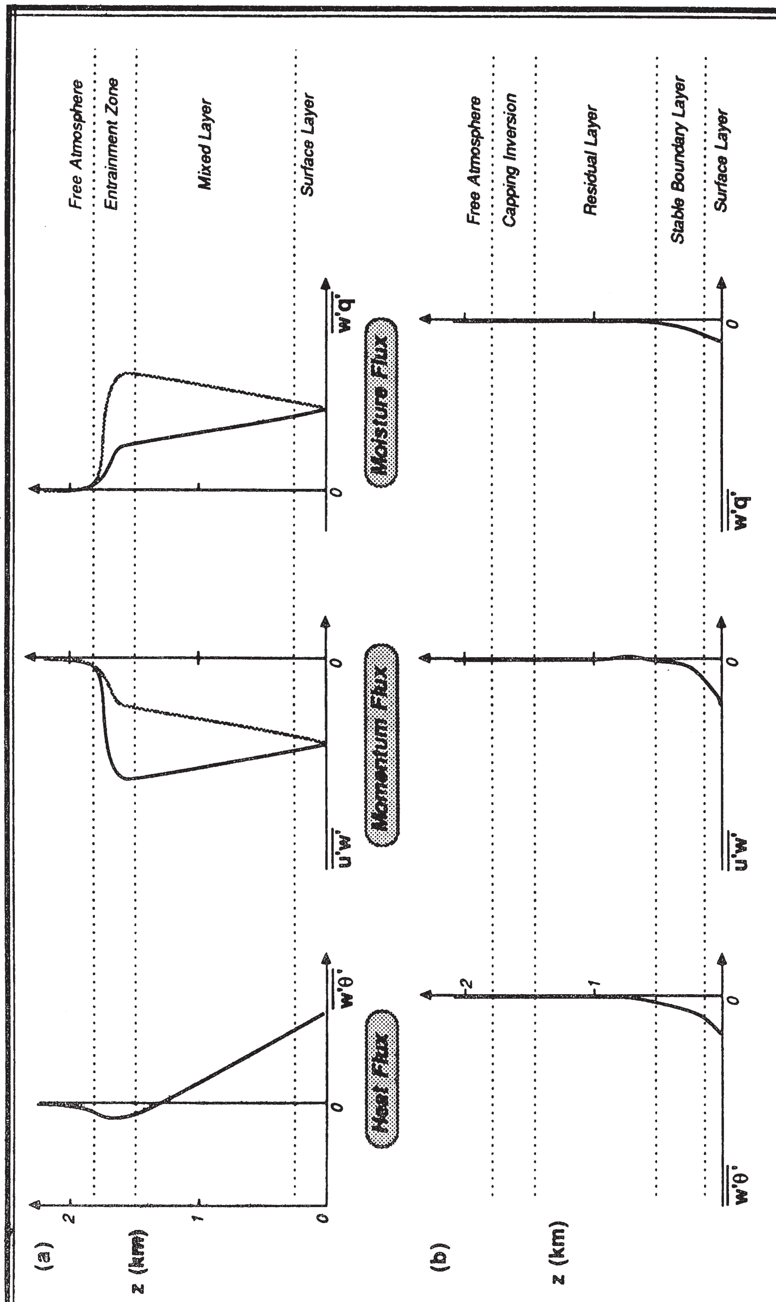


Fig. 2.15 Idealized vertical soundings of turbulent fluxes for the (a) convective mixed layer and (b) stable boundary layer.