Lecture 2: QG Theory / Derivation

① QG vorticity: $$\boxed{\frac{\partial \zeta_g}{\partial t} = \underbrace{-\vec{u}_g \cdot \vec{\nabla}(\zeta_g + f)}_{①} + \underbrace{f_0 \frac{\partial \omega}{\partial p}}_{②}}$$

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}$$

Recall: $\vec{\omega} \doteq \vec{\nabla} \times \vec{u}$ & $\zeta = \hat{k} \times \vec{\omega}$

Recall: $$\frac{d\vec{u}}{dt} = \underbrace{-f\hat{k}\times\vec{v}}_{\text{cor}} \underbrace{-\vec{\nabla}_p \phi}_{\text{PGF}}$$

→ if cor = PGF ⇒ $\frac{d\vec{u}}{dt} = 0$ (geostrophic)

→ if $\frac{d\vec{u}}{dt} \neq 0 \Rightarrow \boxed{\text{QG}}$

Motivation!

① Advection ② Planetary vorticity stretching

② QG Thermo: $$\boxed{\underbrace{\frac{\partial}{\partial t}\left(-\frac{\partial \phi}{\partial p}\right)}_{①} = \underbrace{-\vec{u}_g \cdot \vec{\nabla}\left(-\frac{\partial \phi}{\partial p}\right)}_{②} + \underbrace{\sigma \omega}_{③}}$$

① Local thickness tend.
② Advection
③ Vertical motion & adiabatic temp changes

$$\left[\sigma \doteq -\frac{RT}{p}\frac{\partial \ln \theta}{\partial p} \quad \text{"Static Stability Parameter"}\right]$$

$$\left[d\phi = g\,dz, \ \phi = \text{geopotential} \ [m^2 s^{-2}]\right]$$

Assumes:
- $Q = 0$ (adiabatic)
- frictionless
- constant f plane ($\beta = 0$)

First step: express ζ_g in terms of ϕ

$$\boxed{\zeta_g} = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{\partial}{\partial x}\left(\frac{1}{f_0}\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{1}{f_0}\frac{\partial \phi}{\partial y}\right) = \frac{1}{f_0}\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{f_0}\frac{\partial^2 \phi}{\partial y^2} = \boxed{\frac{1}{f_0}\vec{\nabla}^2 \phi}$$

[insert χ stuff here later if desired!!]

(4) QG ω

(1) QG ζ_g: $$\frac{\partial \zeta_g}{\partial t} = -\vec{u}_g \cdot \vec{\nabla}(\zeta_g + f) + f_0 \frac{\partial \omega}{\partial p} \xRightarrow{\text{plug in}} \frac{\partial}{\partial t}\left(\frac{1}{f_0}\vec{\nabla}^2 \phi\right) = -u_g \cdot \vec{\nabla}\Big(\underbrace{\frac{1}{f_0}\vec{\nabla}^2\phi}_{\zeta_g \text{ (}=\text{)}} + f\Big) + f_0 \frac{\partial \omega}{\partial p}$$

$$\Rightarrow \boxed{\frac{1}{f_0}\vec{\nabla}^2 \frac{\partial \phi}{\partial t} = -\vec{u}_g \cdot \vec{\nabla}\Big(\underbrace{\frac{1}{f_0}\vec{\nabla}^2\phi}_{\zeta_g} + f\Big) + f_0 \frac{\partial \omega}{\partial p}}$$

(2) QG thermo: $$\frac{\partial}{\partial t}\left(-\frac{\partial \phi}{\partial p}\right) = -\vec{u}_g \cdot \vec{\nabla}\left(-\frac{\partial \phi}{\partial p}\right) + \sigma\omega$$

$$\boxed{-\frac{\partial}{\partial p}\left(\frac{\partial \phi}{\partial t}\right) = \vec{u}_g \cdot \vec{\nabla}\left(\frac{\partial \phi}{\partial p}\right) + \sigma\omega}$$

Step 1: take $f_0 \frac{\partial}{\partial p}$(1) $\Rightarrow$ $$f_0 \frac{\partial}{\partial p}\left(\frac{1}{f_0}\vec{\nabla}^2 \frac{\partial \phi}{\partial t}\right) = f_0 \frac{\partial}{\partial p}\Big(-\vec{u}_g \cdot \vec{\nabla}\Big(\overbrace{\frac{1}{f_0}\vec{\nabla}^2\phi}^{\zeta_g} + f\Big)\Big) + f_0 \frac{\partial}{\partial p}\left(f_0 \frac{\partial \omega}{\partial p}\right)$$

(a) $$\frac{\partial}{\partial p}\left(\vec{\nabla}^2 \frac{\partial \phi}{\partial t}\right) = f_0 \frac{\partial}{\partial p}\left(-\vec{u}_g \cdot \vec{\nabla}(\zeta_g + f)\right) + f_0^2 \frac{\partial^2 \omega}{\partial p^2}$$

Step 2: take $\vec{\nabla}^2$(2) $\Rightarrow$ $$\vec{\nabla}^2\left(-\frac{\partial}{\partial p}\left(\frac{\partial \phi}{\partial t}\right)\right) = \vec{\nabla}^2\left(\vec{u}_g \cdot \vec{\nabla}\left(\frac{\partial \phi}{\partial p}\right)\right) + \vec{\nabla}^2(\sigma\omega)$$

(b) $$-\frac{\partial}{\partial p}\left(\vec{\nabla}^2 \frac{\partial \phi}{\partial t}\right) = \vec{\nabla}^2\left(\vec{u}_g \cdot \vec{\nabla}\frac{\partial \phi}{\partial p}\right) + \sigma \vec{\nabla}^2 \omega$$

Step 3: (a) + (b) $\Rightarrow$ (LHS = 0) $\Rightarrow$ $$0 = f_0 \frac{\partial}{\partial p}\left(-\vec{u}_g \cdot \vec{\nabla}(\zeta_g + f)\right) + f_0^2 \frac{\partial^2 \omega}{\partial p^2} + \vec{\nabla}^2\left(\vec{u}_g \cdot \vec{\nabla}\frac{\partial \phi}{\partial p}\right) + \sigma \vec{\nabla}^2 \omega$$

Rearrange...

$$-\sigma \vec{\nabla}^2 \omega - f_0^2 \frac{\partial^2 \omega}{\partial p^2} = f_0 \frac{\partial}{\partial p}\left(-\vec{u}_g \cdot \vec{\nabla}(\zeta_g + f)\right) + \vec{\nabla}^2\left(\vec{u}_g \cdot \vec{\nabla}\left(\frac{\partial \phi}{\partial p}\right)\right)$$

Multiply by -1 and factor...

QG ω $$\boxed{\underbrace{\sigma\left(\vec{\nabla}^2 + \frac{f_0^2}{\partial p^2}\right)\omega}_{①} = \underbrace{-f_0 \frac{\partial}{\partial p}\left[-\vec{u}_g \cdot \vec{\nabla}(\zeta_g + f)\right]}_{②} \underbrace{- \vec{\nabla}^2\left[-\vec{u}_g \cdot \vec{\nabla}\left(-\frac{\partial \phi}{\partial p}\right)\right]}_{③}}$$

(1) $(\dots) \sim$ negative sign
$-\omega = w > 0$

(2) Differential vorticity advection

(3) Temperature advection

QG ω takeaways

① For lower stability and equal forcing (RHS), ω increases

② Increasing PVA w/ height → rising motion

③ WAA → rising motion

Some more notes:

Re: ② ... 1) PVA means $-\vec{u}_g \cdot \vec{\nabla}(\zeta_g + f) > 0$

2) $\frac{\partial(PVA)}{\partial z} > 0$ means $-\frac{\partial}{\partial p}(PVA) > 0$

3) RHS > 0 means LHS > 0, so $\boxed{\omega < 0}$ rising motion

Re: ③ ... 1) WAA means $-\vec{u}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) > 0$ $\left(-\frac{\partial \Phi}{\partial p}\right.$ scales w/ T)

2) $\vec{\nabla}^2$ makes it < 0, and minus sign makes it > 0

3) Same as ② above

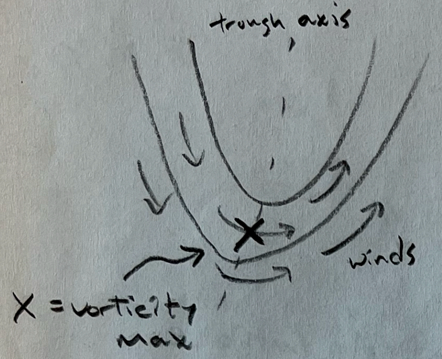
⇒ Now onto examples...

Lecture 2-ish: QG Applications and Intro to Thermal Wind

① Let's look at ω...

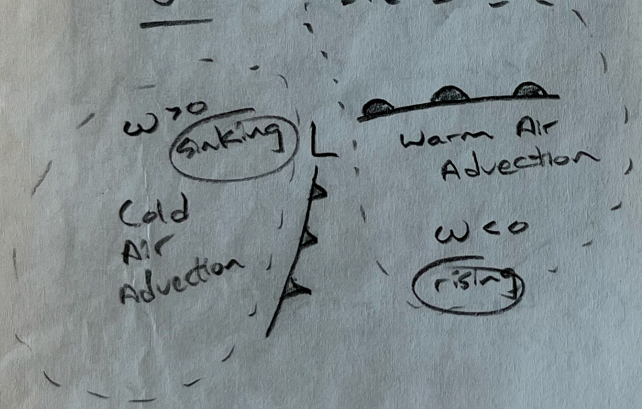
Recall: $\omega \propto$ Temperature Advection & Differential Vorticity Advection

ex) upper-level trough (~500mb), and assume vort. adv. at surface is negligible

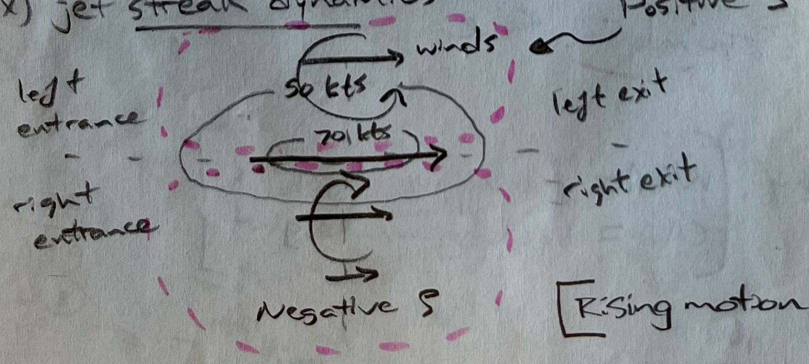


- on west side, negative differential vorticity advection occurs ⇒ $\omega > 0$ (sinking) (DAVA)
- on east side, positive differential vorticity advection occurs ⇒ $\omega < 0$ (rising) (DCVA)

ex) Surface low pressure system



ex) jet streak dynamics



① left entrance: DAVA ⇒ $\omega > 0$

② right entrance: DCVA ⇒ $\omega < 0$

③ left exit: DCVA ⇒ $\omega < 0$

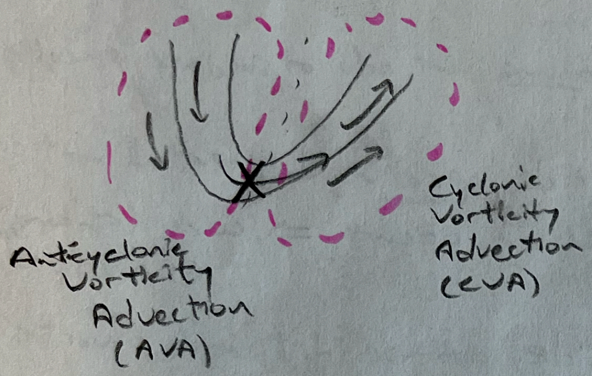
④ right exit: DAVA ⇒ $\omega > 0$

[Rising motion in left exit and right entrance]

② Now let's look at χ.

Recall: $\chi \propto$ Diff'l Temp Adv. & Vort. Advection

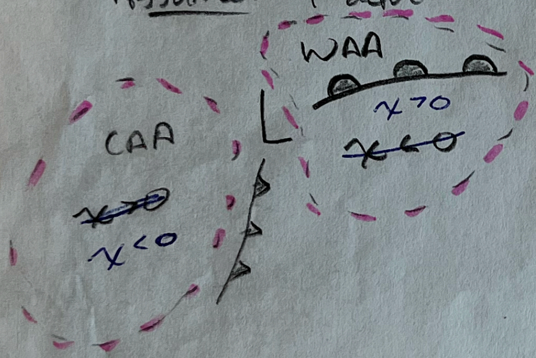
ex) upper-level trough



[Height falls east of the trough.]

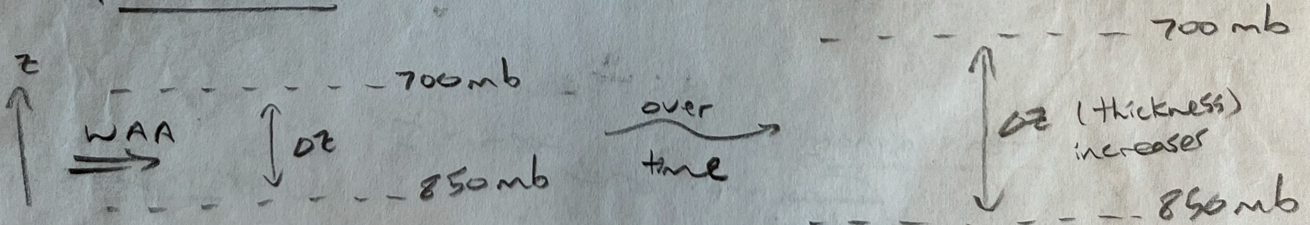
ex) Temperature Advection

Assume: T advection is strongest at the surface



[Trough "digs" into region of surface CAA.]

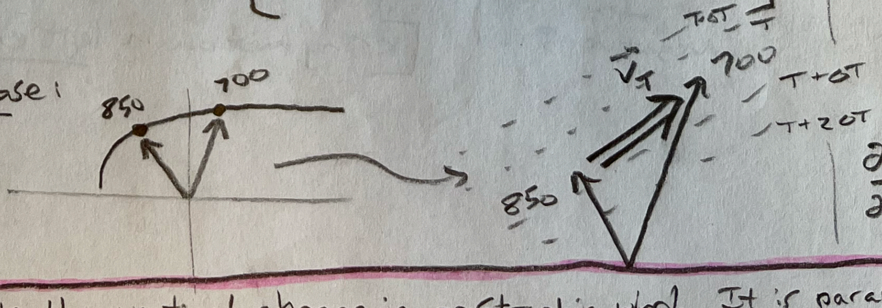
ex) Temp Advection in a layer (& thermal wind)



Assuming geostrophy, [WAA = veering w/ height, &
CAA = backing w/ height]

Assume:
① Geostrophy
② Hydrostatics

WAA case:



$$\frac{\partial \vec{v}_g}{\partial z} = \frac{g}{fT}\hat{k} \times \vec{\nabla}_p T$$

Thermal wind is the vertical change in geostrophic wind. It is parallel to average isotherms in that layer, w/ cold to the left.