Review on Vorticity

Vorticity

- **Vorticity and circulation are both measures of fluid rotation**
  - **Vorticity: measure of rotation at a specific point**
  - **Circulation: measure of rotation over an area**

- **For incompressible fluid, vorticity equation can be expressed as:**

$$\frac{d}{dt}\left(\boldsymbol{\omega}+f\hat{k}\right)=\frac{\partial\boldsymbol{\omega}}{\partial t}+(\boldsymbol{V}\cdot\boldsymbol{\nabla})\left(\boldsymbol{\omega}+f\hat{k}\right)=\left[\left(\boldsymbol{\omega}+f\hat{k}\right)\cdot\boldsymbol{\nabla}\right]\boldsymbol{V}+\frac{1}{\rho^2}\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p+\boldsymbol{\nabla}\times\boldsymbol{F_r}$$

Total vorticity tendency — *local tendency* — *advection* — *tilting and stretching* — *baroclinic generation* — *friction*

$$\boldsymbol{\omega}=(\xi,\eta,\zeta)=\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z},\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x},\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$\boldsymbol{\omega}\equiv 3D\ vector\ vorticity$ not to be confused with the scalar ω, which is vertical velocity in pressure coordinates

- **Baroclinic generation term can be written in terms of buoyancy:**

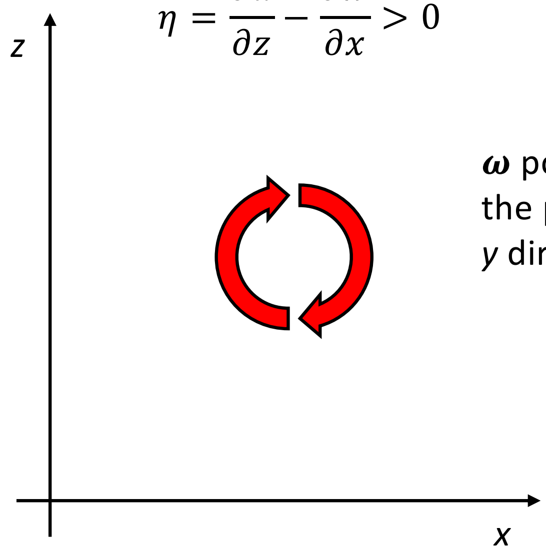
$$\frac{d}{dt}\left(\boldsymbol{\omega}+f\hat{k}\right)=\frac{\partial\boldsymbol{\omega}}{\partial t}+(\boldsymbol{V}\cdot\boldsymbol{\nabla})\left(\boldsymbol{\omega}+f\hat{k}\right)=\left[\left(\boldsymbol{\omega}+f\hat{k}\right)\cdot\boldsymbol{\nabla}\right]\boldsymbol{V}+\boldsymbol{\nabla}\times\boldsymbol{B}\hat{k}+\boldsymbol{\nabla}\times\boldsymbol{F_r}$$

We will revisit baroclinic term later in semester.

Vorticity

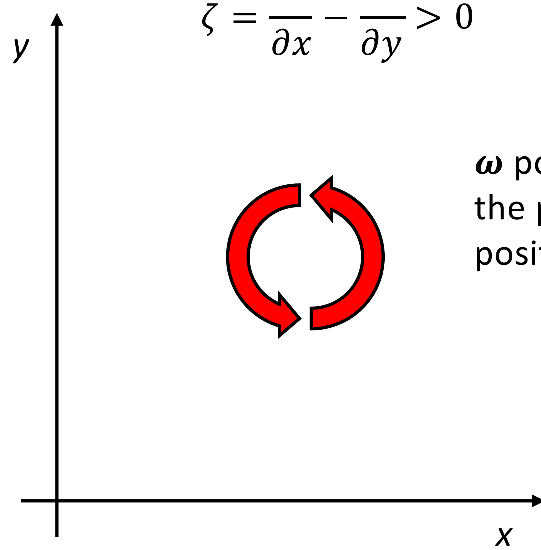
$$\boldsymbol{\omega} = (\xi, \eta, \zeta) = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} > 0$$



$\boldsymbol{\omega}$ points into the page in positive y direction

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} > 0$$



$\boldsymbol{\omega}$ points out of the page in positive z direction

Vertical Vorticity

- **Vertical vorticity tendency is written as (neglecting friction)**

$$\underbrace{\frac{\partial \zeta}{\partial t}}_{\text{local tendency}} = \underbrace{-\boldsymbol{V} \cdot \boldsymbol{\nabla}(\zeta + f)}_{\text{advection}} + \underbrace{\boldsymbol{\omega} \cdot \boldsymbol{\nabla} w}_{\text{tilting and stretching}} + \underbrace{f \frac{\partial w}{\partial z}}_{\text{stretching of planetary vorticity}}$$

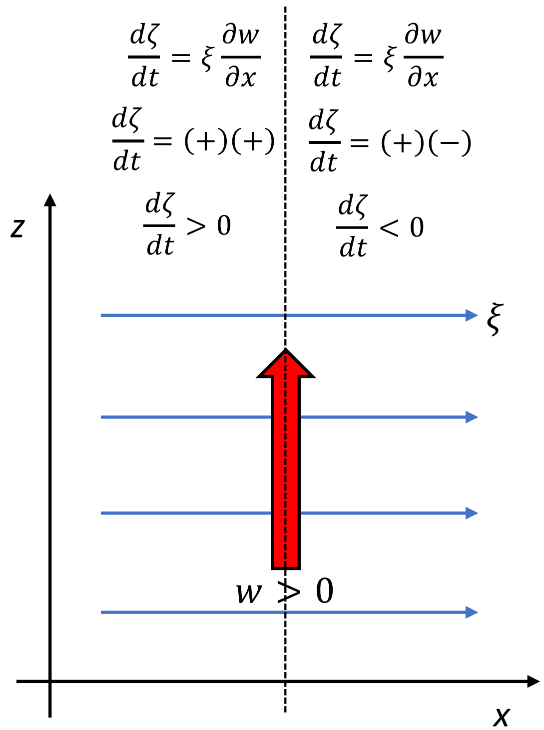
Note that buoyancy does not directly generate vertical vorticity.

- **Lets unpack and interpret the "tilting and stretching" term**

$$\underbrace{\frac{d\zeta}{dt}}_{\text{total tendency}} = \underbrace{\boldsymbol{\omega} \cdot \boldsymbol{\nabla} w}_{\text{tilting and stretching}} = \underbrace{\boxed{\xi \frac{\partial w}{\partial x} + \eta \frac{\partial w}{\partial y}}}_{\text{tilting}} \underbrace{\boxed{+ \zeta \frac{\partial w}{\partial z}}}_{\text{stretching}}$$

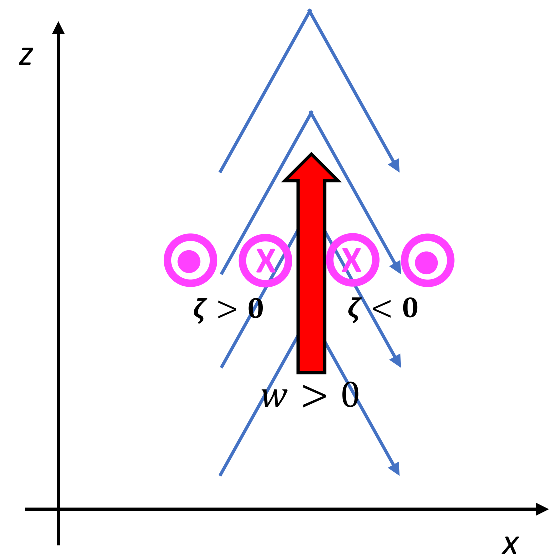
Tilting

$$\xi \frac{\partial w}{\partial x} + \eta \frac{\partial w}{\partial y}$$



time t

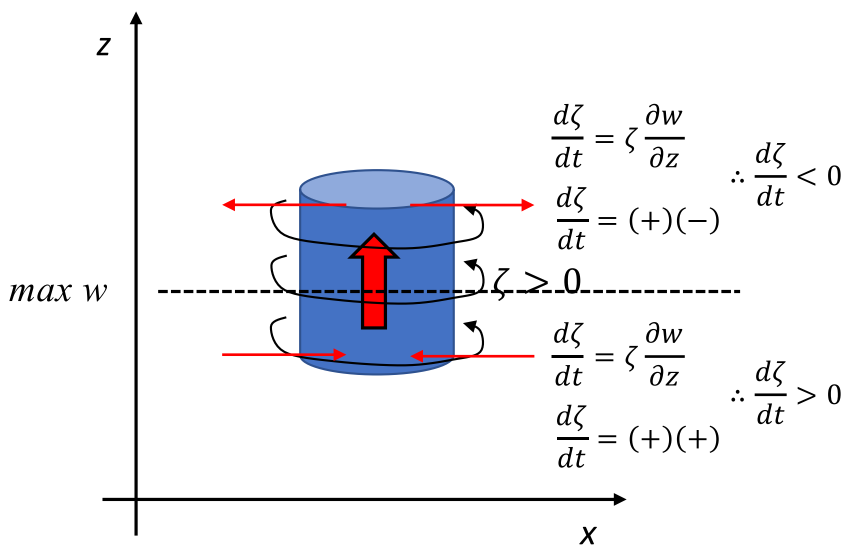
Tilting of horizontal vorticity generates vertical vorticity couplet



time $t + \Delta t$

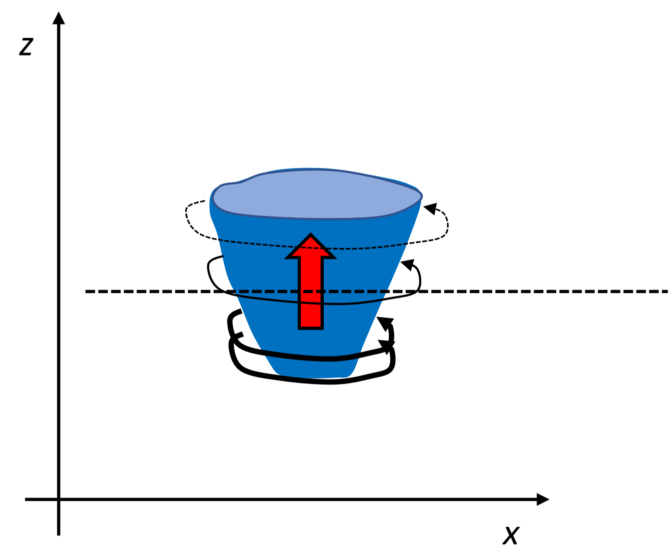
Stretching

$$\zeta \frac{\partial w}{\partial z}$$



time t

Convergence (divergence) spins up (down) vertical vorticity beneath (above) max updraft



time $t + \Delta t$

Origin of Midlevel Rotation in Supercells

Begin with vertical vorticity equation

- **Vertical vorticity equation (neglect Coriolis, baroclinic and friction terms)**

$$\frac{\partial \zeta}{\partial t} = -\boldsymbol{V} \cdot \boldsymbol{\nabla} \zeta + \boldsymbol{\omega} \cdot \boldsymbol{\nabla} w$$

advection — tilting and stretching

- **Expand**

$$\frac{\partial \zeta}{\partial t} = -u \frac{\partial \zeta}{\partial x} - v \frac{\partial \zeta}{\partial y} - w \frac{\partial \zeta}{\partial z} + \xi \frac{\partial w}{\partial x} + \eta \frac{\partial w}{\partial y} + \zeta \frac{\partial w}{\partial z}$$

horizontal advection — vertical advection — tilting — stretching

- **Expand fully**

$$\frac{\partial \zeta}{\partial t} = -u \frac{\partial \zeta}{\partial x} - v \frac{\partial \zeta}{\partial y} - w \frac{\partial \zeta}{\partial z} + \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \frac{\partial w}{\partial x} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \frac{\partial w}{\partial y} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \frac{\partial w}{\partial z}$$

horizontal advection — vertical advection — tilting — stretching

Linearize vertical vorticity equation around an environmental wind profile $[\bar{u}(z), \bar{v}(z)]$

- **Linearize and neglect product of perturbation terms**

$$u = \bar{u}(z) + u' \qquad w = w'$$
$$v = \bar{v}(z) + v' \qquad \zeta = \zeta'$$

$$\frac{\partial \zeta}{\partial t} = -u\frac{\partial \zeta}{\partial x} - v\frac{\partial \zeta}{\partial y} - w\frac{\partial \zeta}{\partial z} + \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\frac{\partial w}{\partial x} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\frac{\partial w}{\partial y} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{\partial w}{\partial z}$$

$$\frac{\partial \zeta'}{\partial t} = \underbrace{-\bar{u}\frac{\partial \zeta'}{\partial x} - \bar{v}\frac{\partial \zeta'}{\partial y}}_{\text{advection}} + \underbrace{\frac{\partial \bar{u}}{\partial z}\frac{\partial w'}{\partial y} - \frac{\partial \bar{v}}{\partial z}\frac{\partial w'}{\partial x}}_{\text{tilting}}$$

$$\frac{\partial \zeta'}{\partial t} = -\bar{\boldsymbol{V}} \cdot \boldsymbol{\nabla_h}\zeta' + \boldsymbol{S}\times\boldsymbol{\nabla_h}w' \cdot \hat{k}$$

$\bar{\boldsymbol{V}}$: mean velocity vector (horizontal wind only)

$\boldsymbol{S} = \frac{\partial \bar{\boldsymbol{V}}}{\partial z}$ mean vertical shear

$$\boldsymbol{S}\times\boldsymbol{\nabla_h}w' = \left(\frac{\partial \bar{u}}{\partial z}\vec{\imath} + \frac{\partial \bar{v}}{\partial z}\vec{\jmath}\right) \times \left(\frac{\partial w'}{\partial x}\vec{\imath} + \frac{\partial w'}{\partial y}\vec{\jmath}\right)$$
$$= \frac{\partial \bar{u}}{\partial z}\frac{\partial w'}{\partial y}\vec{k} - \frac{\partial \bar{v}}{\partial z}\frac{\partial w'}{\partial x}\vec{k}$$

- **Rewritten in reference frame moving with storm**

$$\frac{\partial \zeta'}{\partial t} = \underbrace{-\overline{(\boldsymbol{V} - \boldsymbol{c})} \cdot \boldsymbol{\nabla_h}\zeta'}_{\text{advection}} + \underbrace{\boldsymbol{S}\times\boldsymbol{\nabla_h}w' \cdot \hat{k}}_{\text{tilting}}$$

Comments on linearized equation

$$\frac{\partial \zeta'}{\partial t} = -\overline{(\boldsymbol{V} - \boldsymbol{c})} \cdot \boldsymbol{\nabla}_{\boldsymbol{h}} \zeta' + \boldsymbol{S} \times \boldsymbol{\nabla}_{\boldsymbol{h}} w' \cdot \hat{k}$$

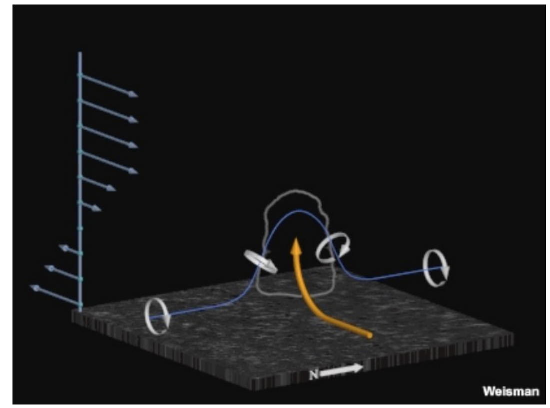
- **Mesocyclone acquires vertical vorticity by tilting horizontal vorticity associated with environment vertical wind shear**

- **Advection shifts location of vorticity around in important ways**

- **Only tilting term can develop $\zeta' > 0$ because advection term is 0 if $\zeta' = 0$ everywhere (in the environment)**

- **As an updraft intensifies, $\zeta' > 0$ initially generated by tilting is intensified by stretching ($\zeta' \frac{\partial w'}{\partial z}$; nonlinear term removed by linearization)**

Interpretation of tilting term $S \times \nabla_h w' \cdot \hat{k}$



- **Tilting term produced vorticity couplet that straddles the updraft core**
- **Negative vorticity is located left of the shear vector**
- **Positive vorticity is located right of the shear vector**

$$\frac{\partial \zeta'}{\partial t} = \frac{\partial \bar{u}}{\partial z}\frac{\partial w'}{\partial y} - \frac{\partial \bar{v}}{\partial z}\frac{\partial w'}{\partial x}$$

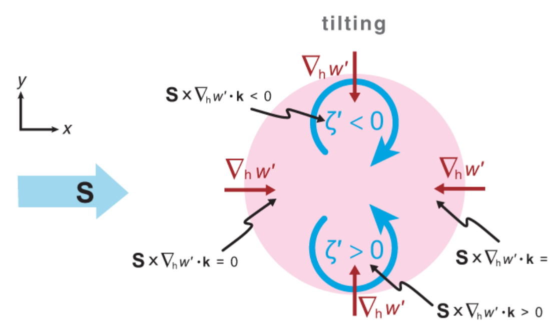
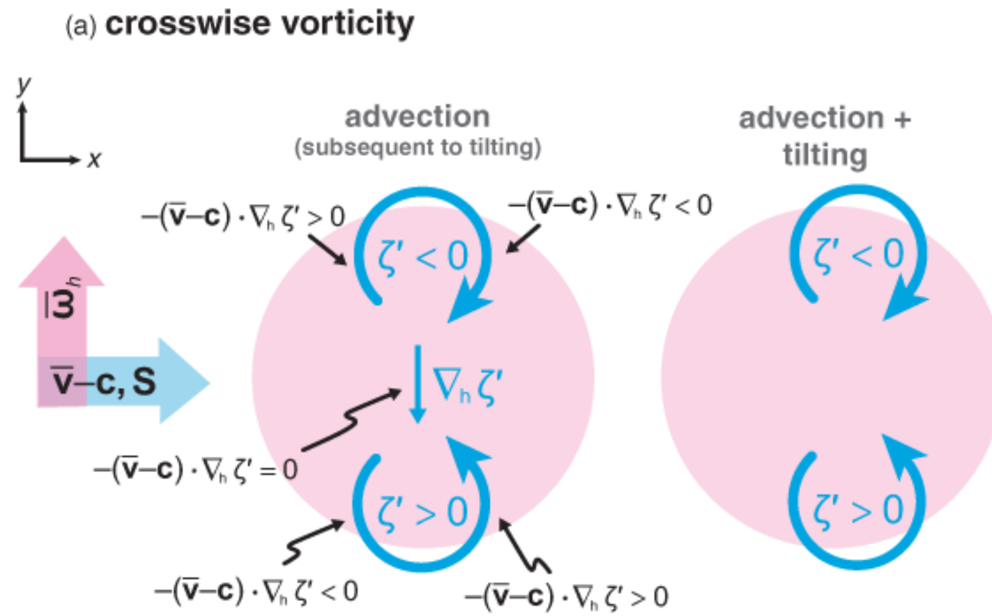


Figure 8.29

Evaluation of $(\partial \zeta'/\partial t)_{sr}$ via the tilting term $(\mathbf{S} \times \nabla_h w' \cdot \mathbf{k})$ in (8.8). The illustration shows the horizontal cross-section of an updraft (pink) at midlevels, indicating the relationship among ζ' (the sense of rotation is given by the curved blue arrows), **S**, and $\nabla_h w'$. Source: MR (2010)

Interpretation of advection term $-\overline{(\boldsymbol{V}-\boldsymbol{c})} \cdot \boldsymbol{\nabla}_h \zeta'$

- **Horizontal vorticity vector ($\boldsymbol{\omega}_h$) points 90° to left of shear vector ($\boldsymbol{S}$)**
- **Crosswise vorticity case**
  - $(\boldsymbol{V}-\boldsymbol{c}) \perp \boldsymbol{\omega}_h$
  - **Advection term does not shift cyclonic vortex toward updraft core**



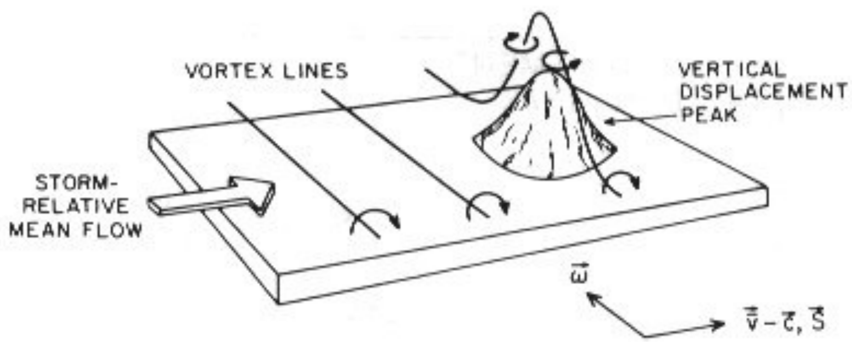


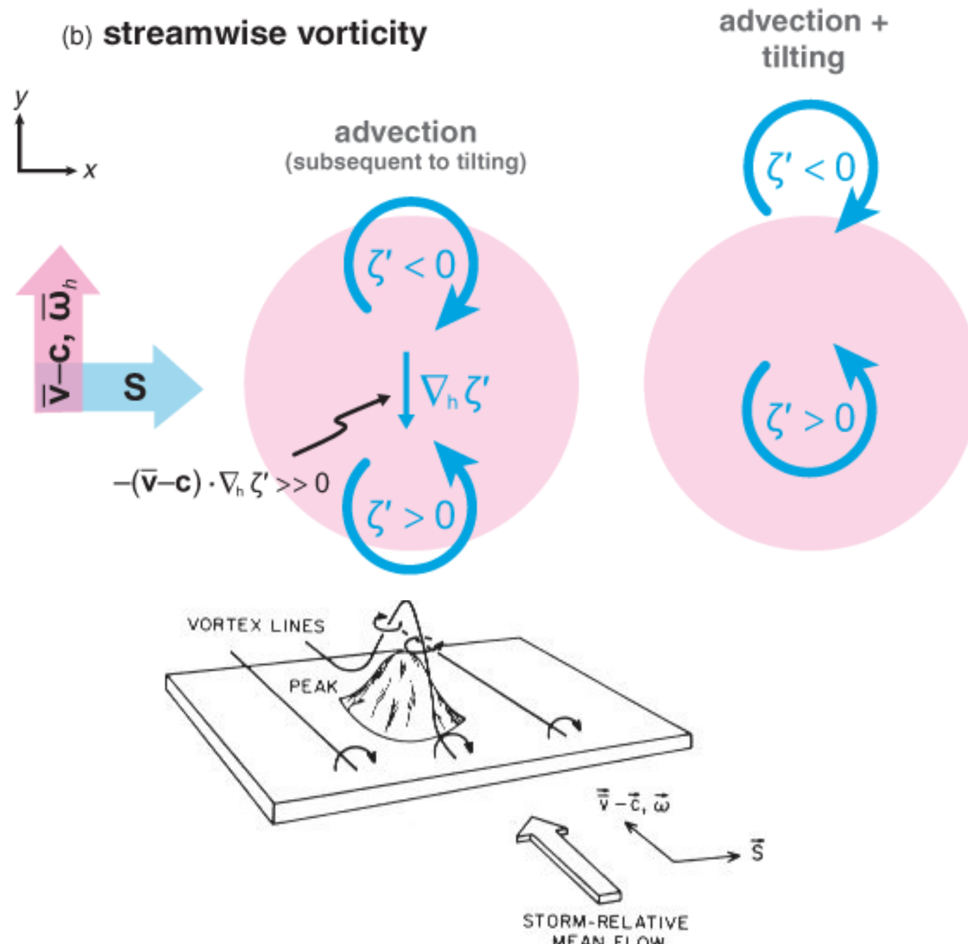
Figure 8.30

Evaluation of $(\partial\zeta'/\partial t)_{sr}$ via the advection term $(-(\bar{\mathbf{v}}-\mathbf{c})\cdot\nabla_h\zeta')$ in (8.8), based on the fact that the tilting term leads to a couplet of vertical vorticity straddling the updraft and oriented normal to the shear vector (cf. Figure 8.29), for the cases of (a) crosswise vorticity and (b) streamwise vorticity. The relationship between the vertical vorticity and vertical velocity fields resulting from the sum of the advection and tilting terms is also shown. As in Figure 8.29, the illustration shows the horizontal cross-section of an updraft (pink) at midlevels and the relationship among ζ' (its sense is given by the curved blue arrows), $\mathbf{S}$, $\bar{\omega}_h$, $\bar{\mathbf{v}}-\mathbf{c}$, and $\nabla_h\zeta'$ The lateral shifting of the ζ' field within the updraft cross-section depends on the orientation of $\bar{\mathbf{v}}-\mathbf{c}$ with respect to $\bar{\omega}_h$ (and $\nabla_h\zeta'$, which points in the opposite direction to $\bar{\omega}_h$). This orientation is related to whether the environmental horizontal vorticity is streamwise or crosswise.

Source: MR (2010)

Interpretation of advection term $-\overline{(\boldsymbol{V}-\boldsymbol{c})}\cdot\boldsymbol{\nabla}_h\zeta'$

- **Horizontal vorticity vector ($\boldsymbol{\omega_h}$) points 90° to left of shear vector ($\boldsymbol{S}$)**
- **Streamwise vorticity case**
  - $(\boldsymbol{V}-\boldsymbol{c}) \parallel \boldsymbol{\omega_h}$
  - **Advection term shifts cyclonic vortex toward updraft, aligning vorticity with updraft core, resulting in cyclonic, intense mesocyclone**
  - **Stretching ($\zeta'\frac{\partial w'}{\partial z}$, nonlinear term) can intensify ζ'**



Source: MR (2010)