Basic Equations

Momentum equations:

$$\frac{du}{dt} - fv = -\frac{\partial \Phi}{\partial x}$$

$$\frac{dv}{dt} + fu = -\frac{\partial \Phi}{\partial y}$$

Here the total derivative is defined as

$$\frac{d}{dt} \equiv \left(\frac{\partial}{\partial t}\right)_p + u\left(\frac{\partial}{\partial x}\right)_p + v\left(\frac{\partial}{\partial y}\right)_p + \omega\frac{\partial}{\partial p}$$

$$\omega = \frac{dp}{dt}$$

Hydrostatic equation:

$$\frac{\partial \Phi}{\partial p} = -\alpha = -\frac{RT}{p}$$

Mass continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

$$\sigma \equiv -\alpha\frac{\partial ln\theta}{\partial p}$$

Is a static stability parameter

$$\theta = T\left(\frac{p_0}{p}\right)^{R/C_p} = T\pi$$

Thermodynamic energy equation:

$$\frac{d\theta}{dt} = \frac{\pi}{C_p}q \rightarrow \left(\frac{\partial \theta}{\partial t}\right)_p + u\left(\frac{\partial \theta}{\partial x}\right)_p + v\left(\frac{\partial \theta}{\partial y}\right)_p + \omega\frac{\partial \theta}{\partial p} = \frac{\pi}{C_p}q \rightarrow \pi\frac{\partial T}{\partial t} + \pi u\frac{\partial T}{\partial x} + \pi v\frac{\partial T}{\partial y} + \omega\frac{\partial \theta}{\partial p} = \frac{\pi}{C_p}q$$

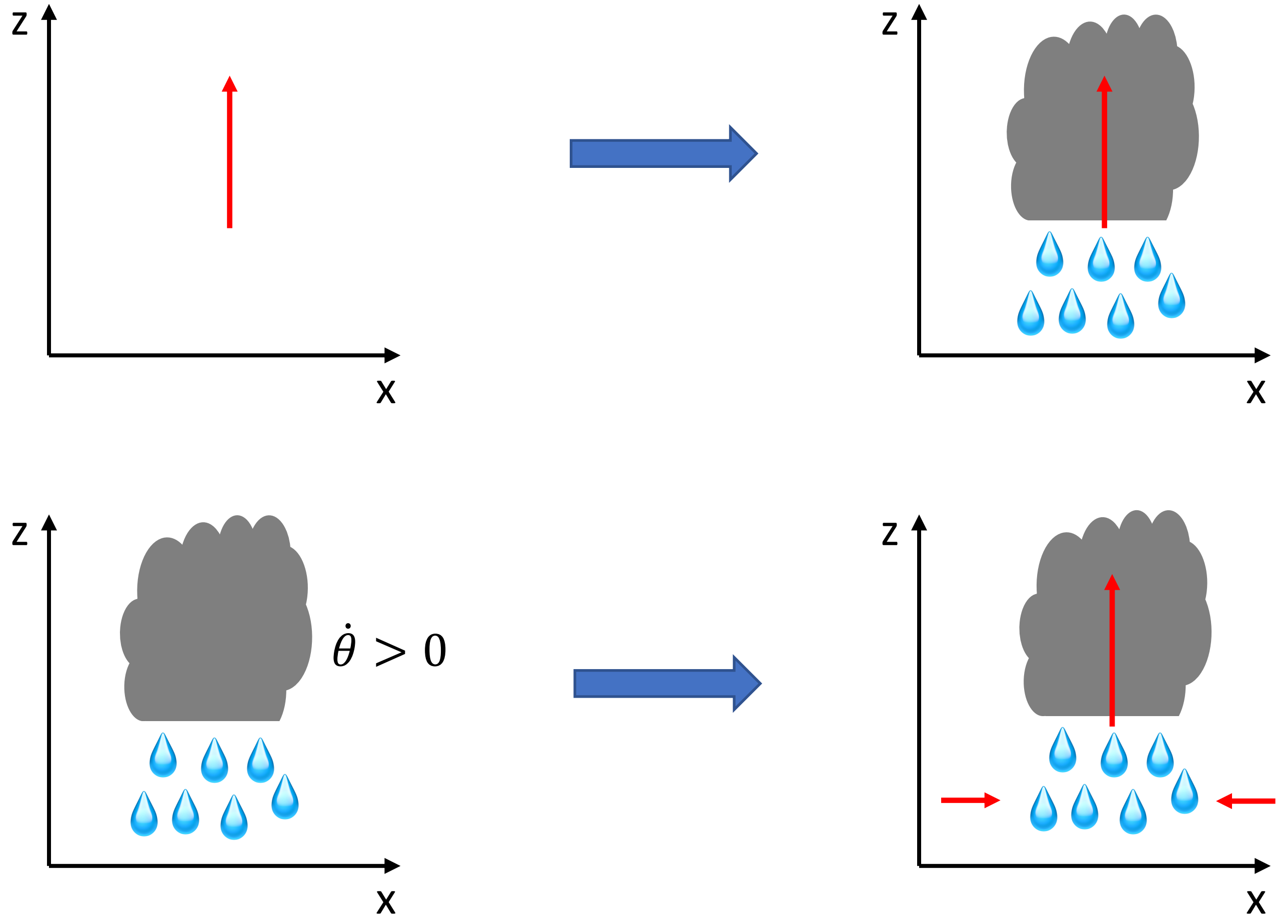
$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \frac{T\omega}{\pi T}\frac{\partial \theta}{\partial p} = \frac{q}{C_p} \rightarrow \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega T\frac{\partial ln\theta}{\partial p} = \frac{q}{C_p} \rightarrow \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - \frac{p}{R}\omega\sigma = \frac{q}{C_p}$$

QG Omega Equation: Physical Interpretation of Terms

Vertical Motion

Midlatitudes:

Tropics:



- Use momentum, vorticity, and thermodynamic equations in quasigeostrophic (QG) framework to obtain diagnostic equation for vertical motion (ω)

QG Omega Equation (Traditional form)

$$\frac{d(\zeta_g + f)}{dt} = f_0 \frac{\partial \omega}{\partial p}$$ QG vorticity equation

$$\frac{dT}{dt} = \frac{p}{R_d}\sigma\omega$$ QG thermodynamic equation

Here $\frac{d}{dt} \equiv \left(\frac{\partial}{\partial t}\right)_p + u_g\left(\frac{\partial}{\partial x}\right)_p + v_g\left(\frac{\partial}{\partial y}\right)_p = \left(\frac{\partial}{\partial t}\right)_p - \boldsymbol{V_g} \cdot \boldsymbol{\nabla_p}(\,)$

- Assume thermal wind balance is maintained locally
- Remove time derivatives
- Combine

See Bluestein Vol1 p329.

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\boldsymbol{V_g} \cdot \boldsymbol{\nabla_p}(\zeta_g + f)\right] - \frac{R_d}{\sigma p}\nabla_p^2\left(-\boldsymbol{V_g} \cdot \boldsymbol{\nabla_p} T\right)$$ QG omega equation (traditional form)

- 2nd derivative in space of ω
- Proportional to negative of ω
- RHS (“forcing”) positive, ω<0 (ascent)
- RHS (“forcing”) negative, ω>0 (descent)

- Laplacian of temperature advection by geostrophic wind
- Warm advection, ω<0 (ascent)
- Cold advection, ω>0 (descent)

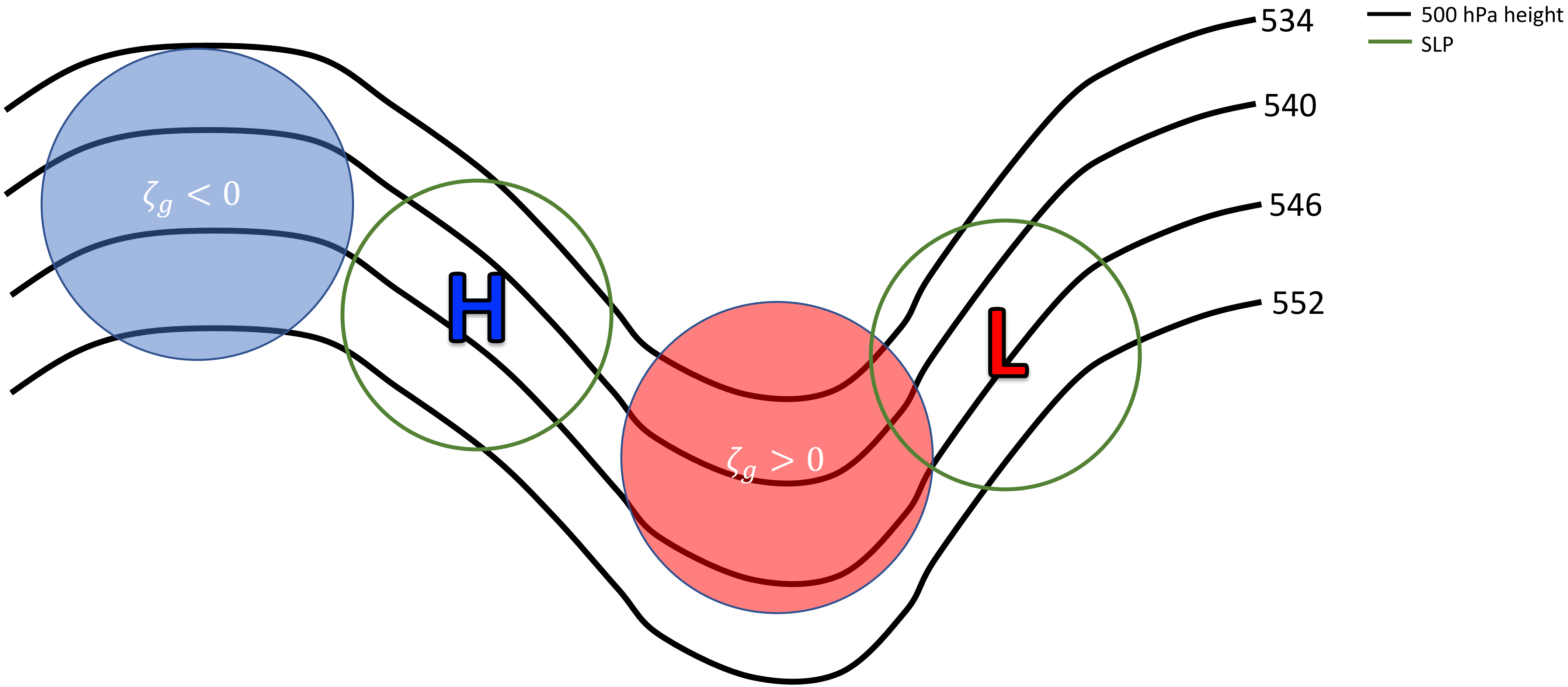
- Differential advection of geostrophic absolute vorticity by the geostrophic wind
- Cyclonic vorticity advection increasing with height, ω<0 (ascent)
- Anticyclonic vorticity advection increasing with height, ω>0 (descent)

$$\omega = \sin(\pi p/p_0)\sin(kx)\sin(ly)$$

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \simeq -\left[(k^2 + l^2) + \frac{f_0^2}{\sigma}\frac{\pi^2}{p_0^2}\right]\omega$$

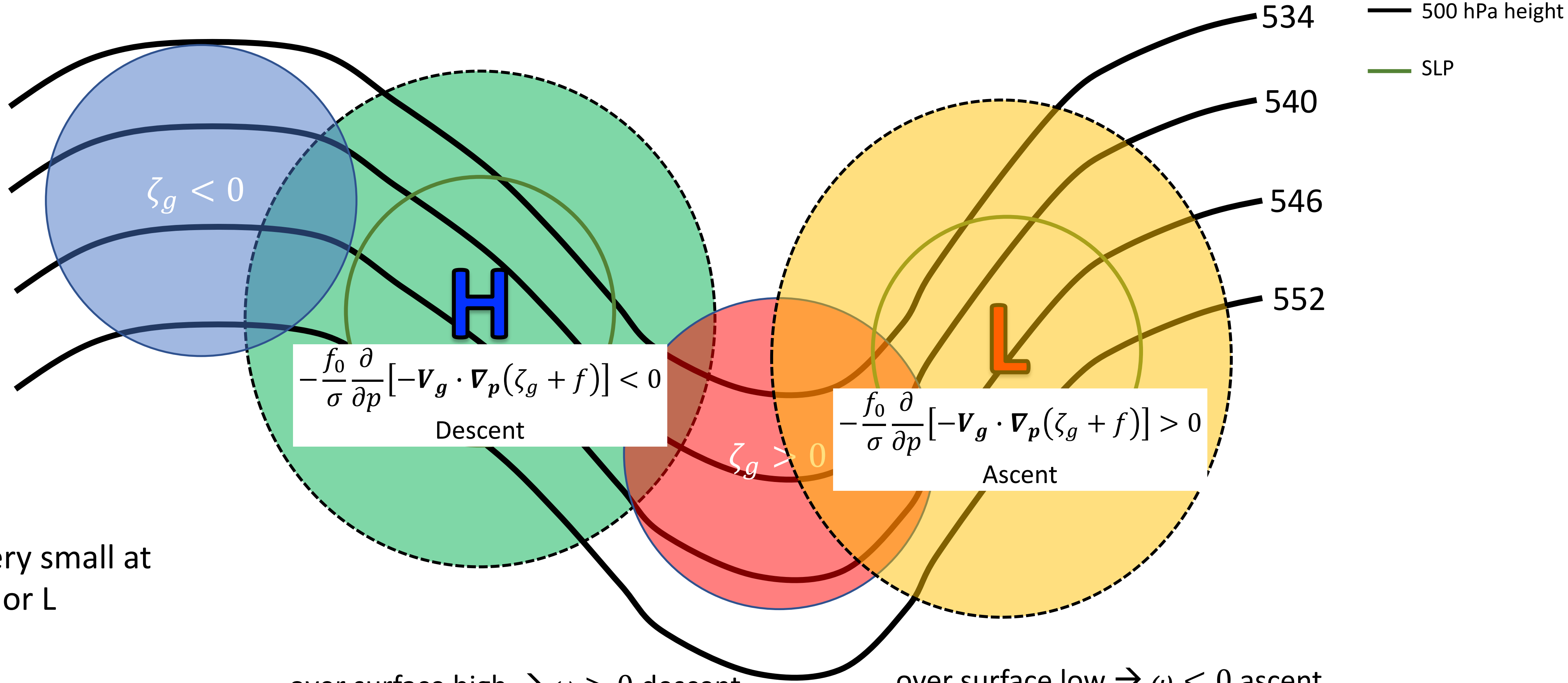
QG Omega Equation: differential vorticity advection term

$$-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\boldsymbol{V_g}\cdot\boldsymbol{\nabla_p}(\zeta_g + f)\right]$$ Differential advection of geostrophic absolute vorticity by the geostrophic wind



QG Omega Equation: differential vorticity advection term

$-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\boldsymbol{V_g}\cdot\boldsymbol{\nabla_p}\left(\zeta_g+f\right)\right]$ Differential advection of geostrophic absolute vorticity by the geostrophic wind



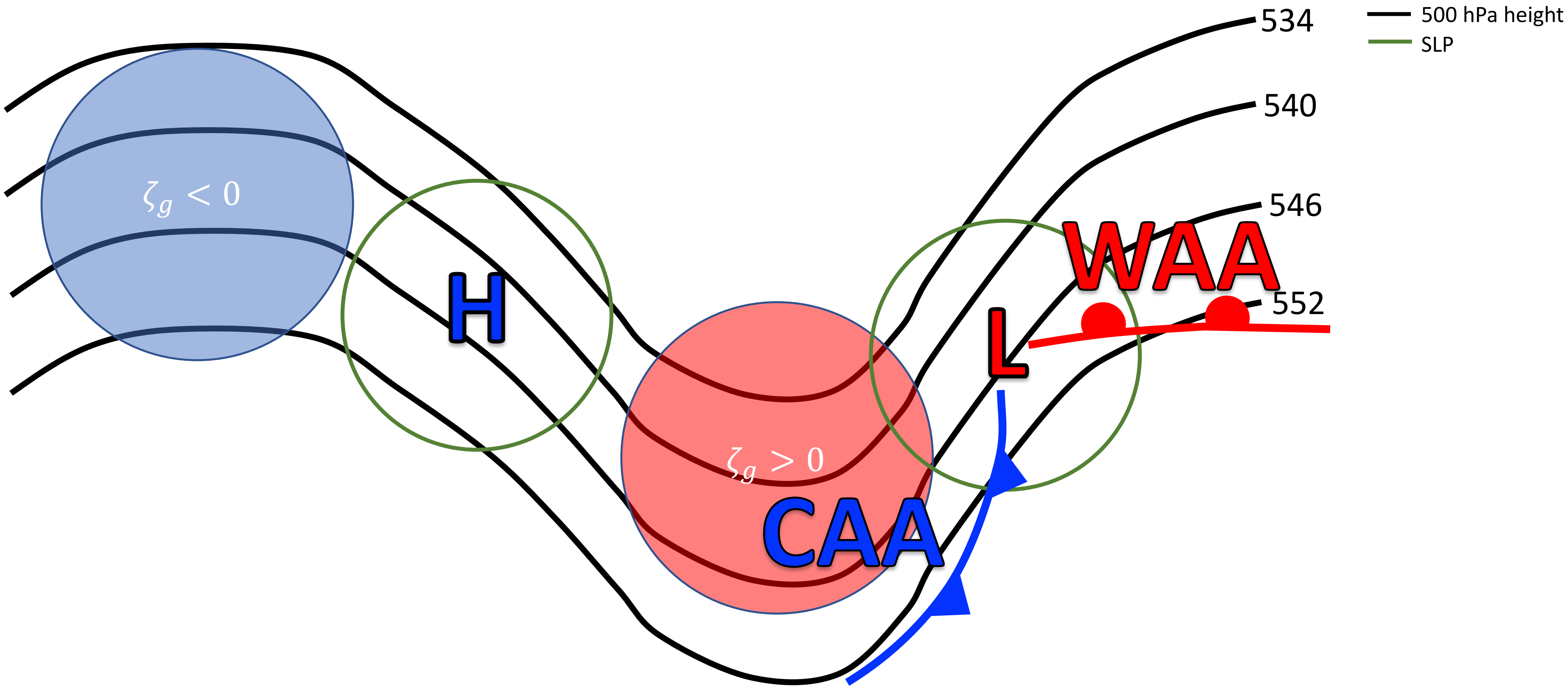
Advection is very small at center of sfc H or L

over surface high → $\omega > 0$ descent

over surface low → $\omega < 0$ ascent

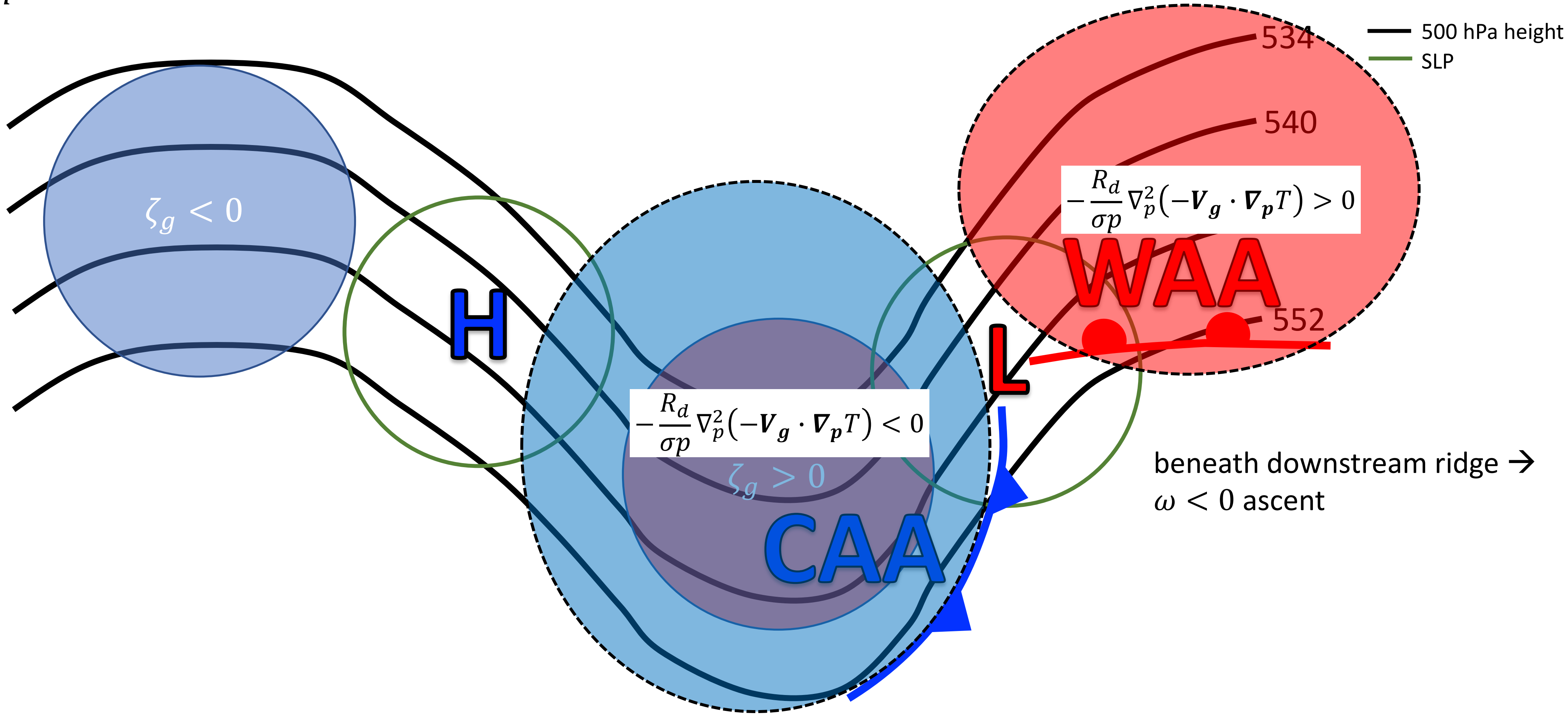
QG Omega Equation: thermal advection term

$-\frac{R_d}{\sigma p}\nabla_p^2\left(-\boldsymbol{V_g}\cdot\boldsymbol{\nabla_p}T\right)$ Laplacian of temperature advection by the geostrophic wind



QG Omega Equation: thermal advection term

$-\frac{R_d}{\sigma p}\nabla_p^2(-\boldsymbol{V_g}\cdot\boldsymbol{\nabla_p}T)$ Laplacian of temperature advection by the geostrophic wind



beneath downstream ridge → $\omega < 0$ ascent

beneath trough → $\omega > 0$ descent

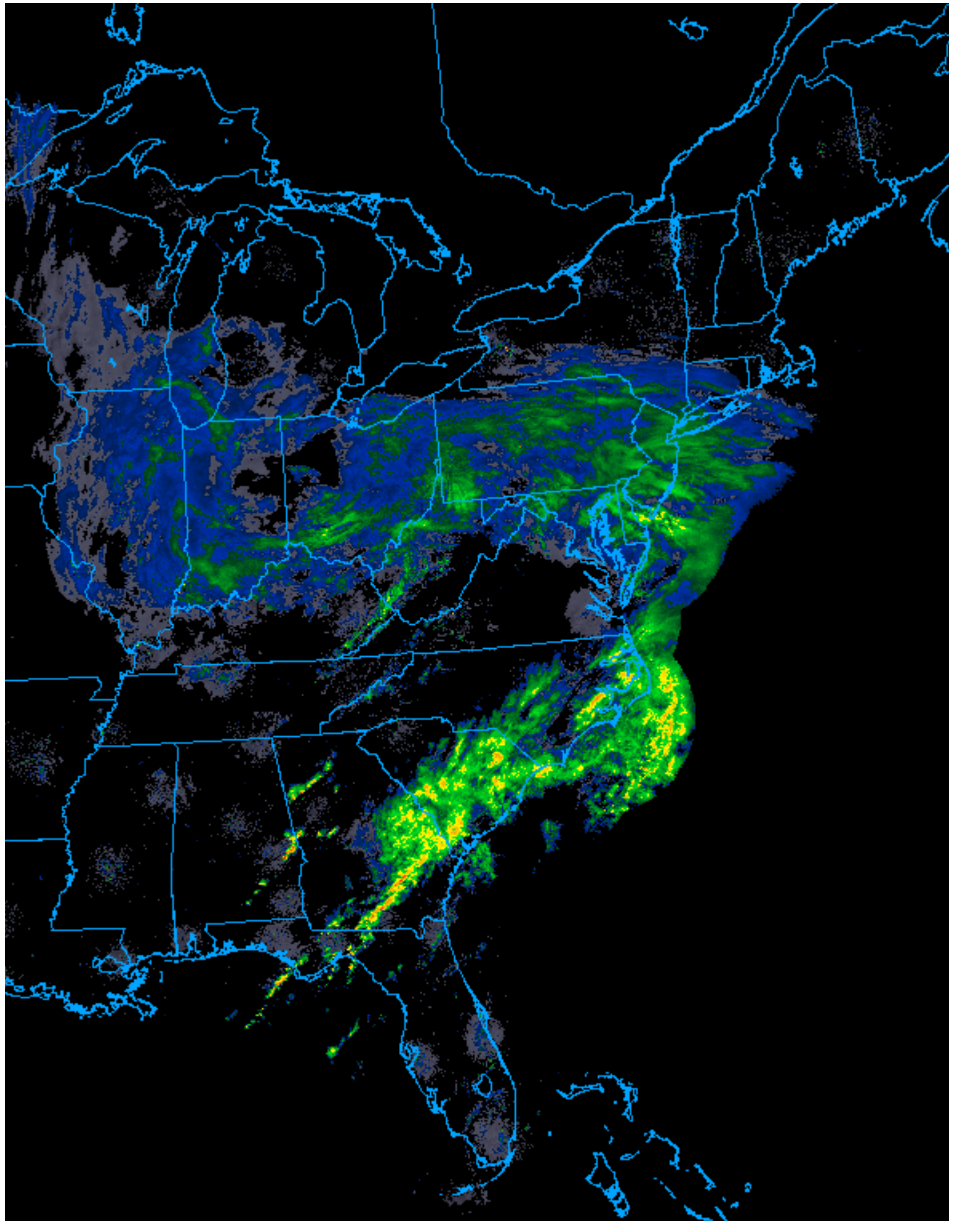
QG Omega Equation (Summary)

$$\underbrace{\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega}_{A} = \underbrace{-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\boldsymbol{V_g}\cdot\boldsymbol{\nabla_p}(\zeta_g + f)\right]}_{B} \underbrace{- \frac{R_d}{\sigma p}\nabla_p^2\left(-\boldsymbol{V_g}\cdot\boldsymbol{\nabla_p}T\right)}_{C}$$

See Bluestein Vol1 p329.

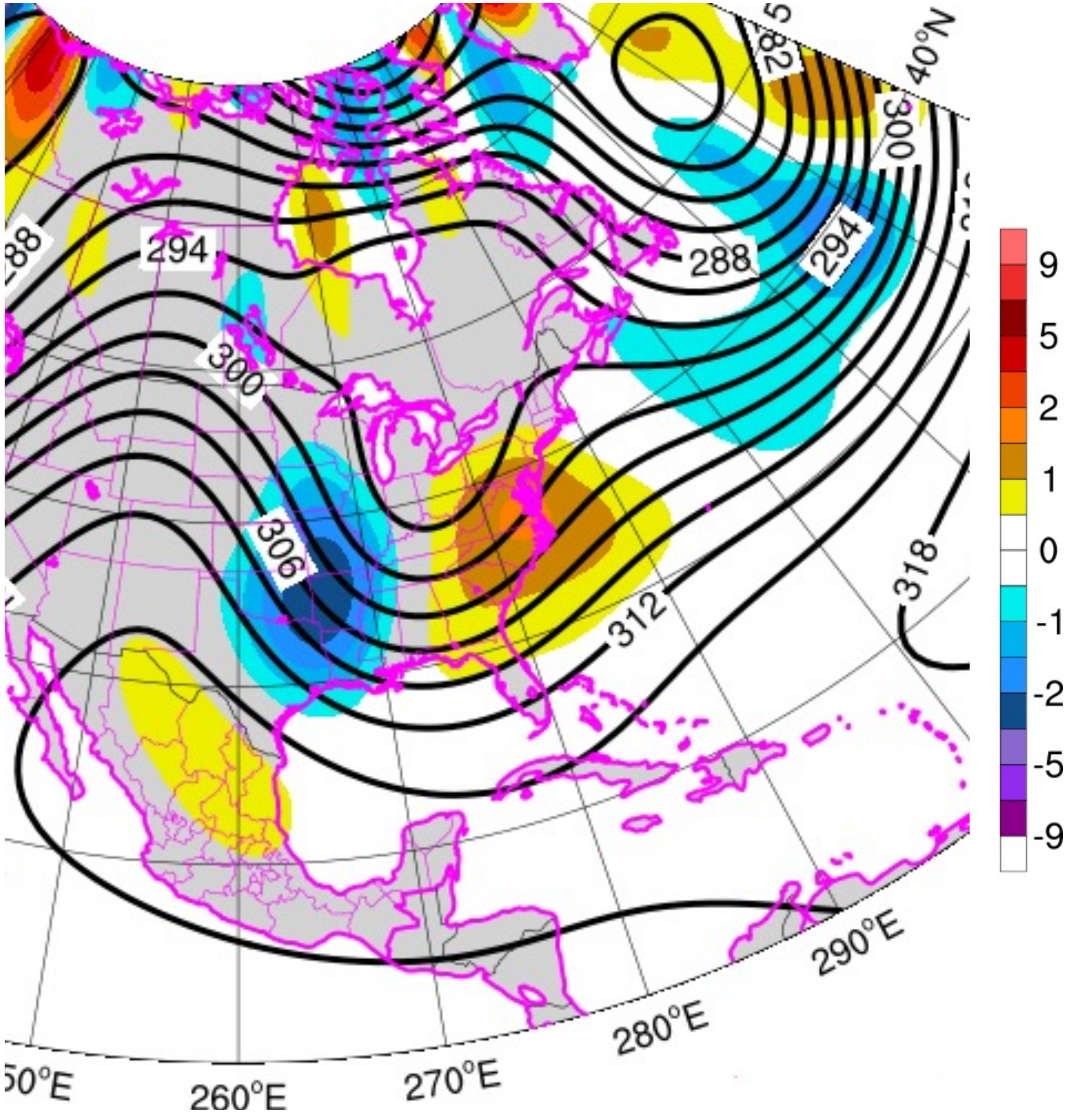
- **Vertical motion (A) = B + C**
- **Term B: differential advection of geostrophic absolute vorticity by the geostrophic wind**
  - **Cyclonic vorticity advection increasing with height ≡ rising motion**
  - **Cyclonic vorticity advection decreasing with height ≡ sinking motion**
- **Term C: Laplacian of temperature advection by the geostrophic wind**
  - **Rising motion with warm advection**
  - **Sinking motion with cold advection**
- **Both terms are scaled by static stability: stronger response with steeper lapse rates (weaker static stability)**

Reflectivity Mosaic
2355Z/31 Jan 2021

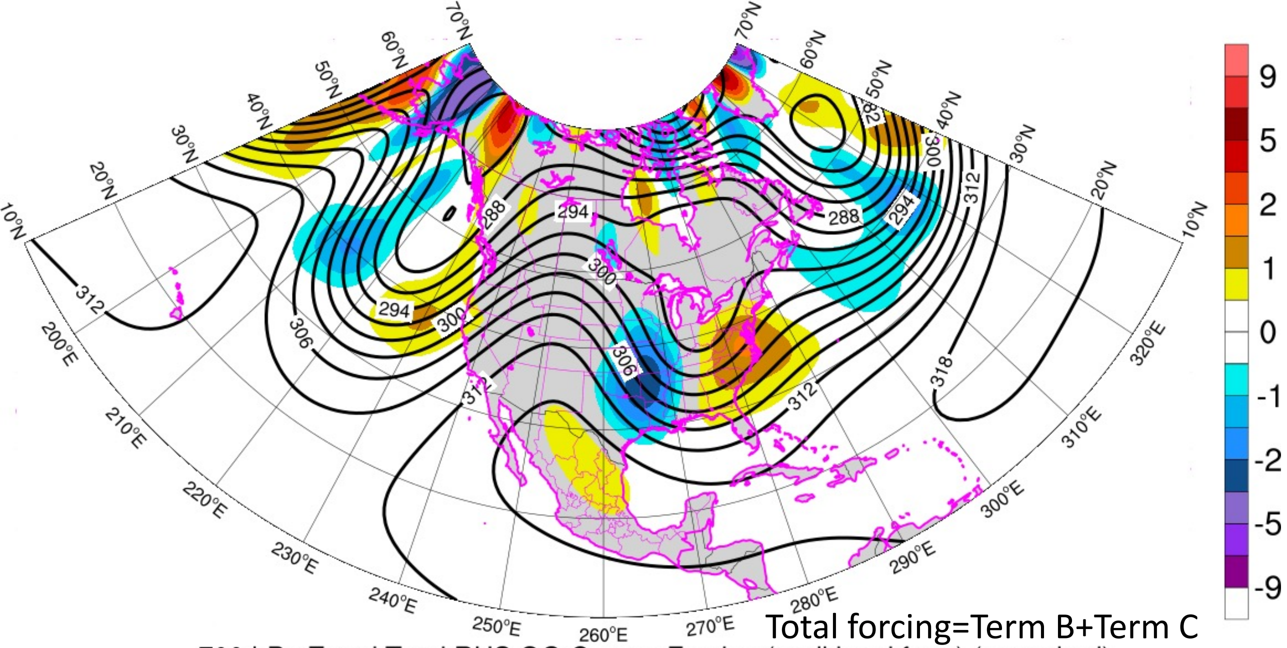


https://www2.mmm.ucar.edu/imagearchive/

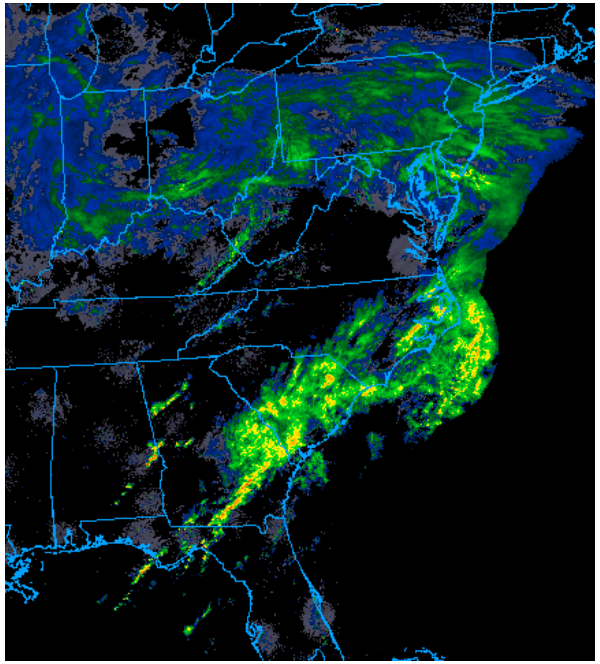
700 hPa Z and QG Omega Forcing (Term B+Term C)
00Z/1 Feb 2021



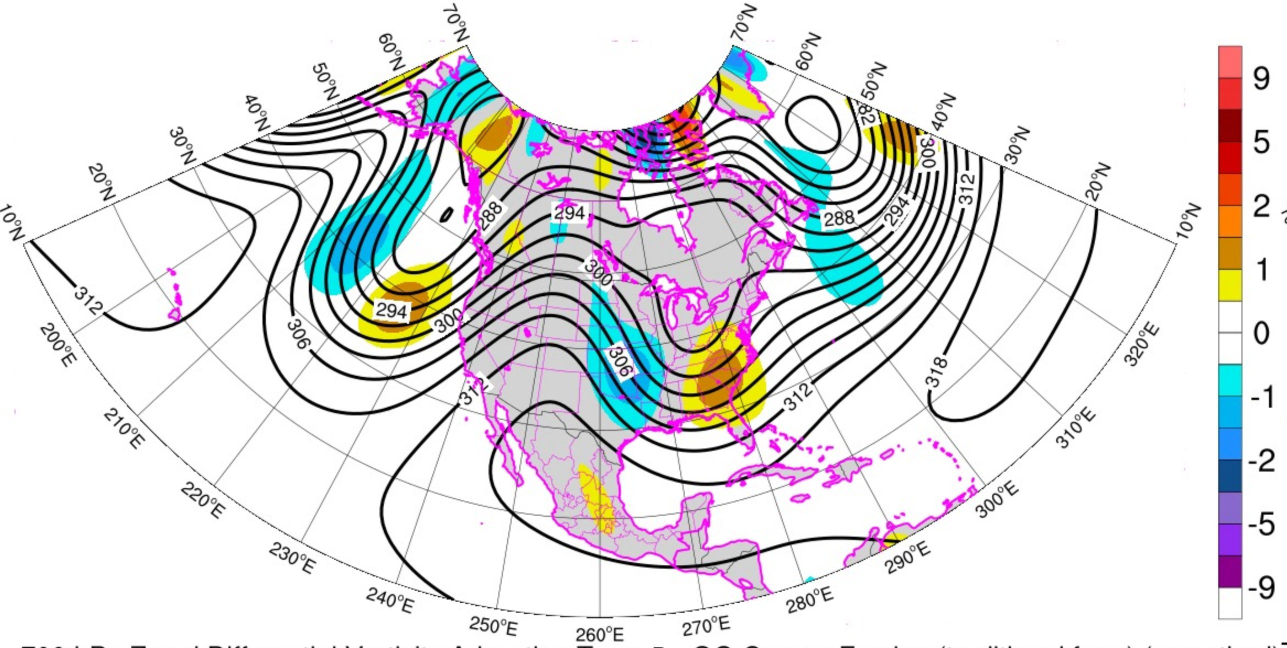
https://inside.nssl.noaa.gov/tgalarneau/real-time-qg-diagnostics/



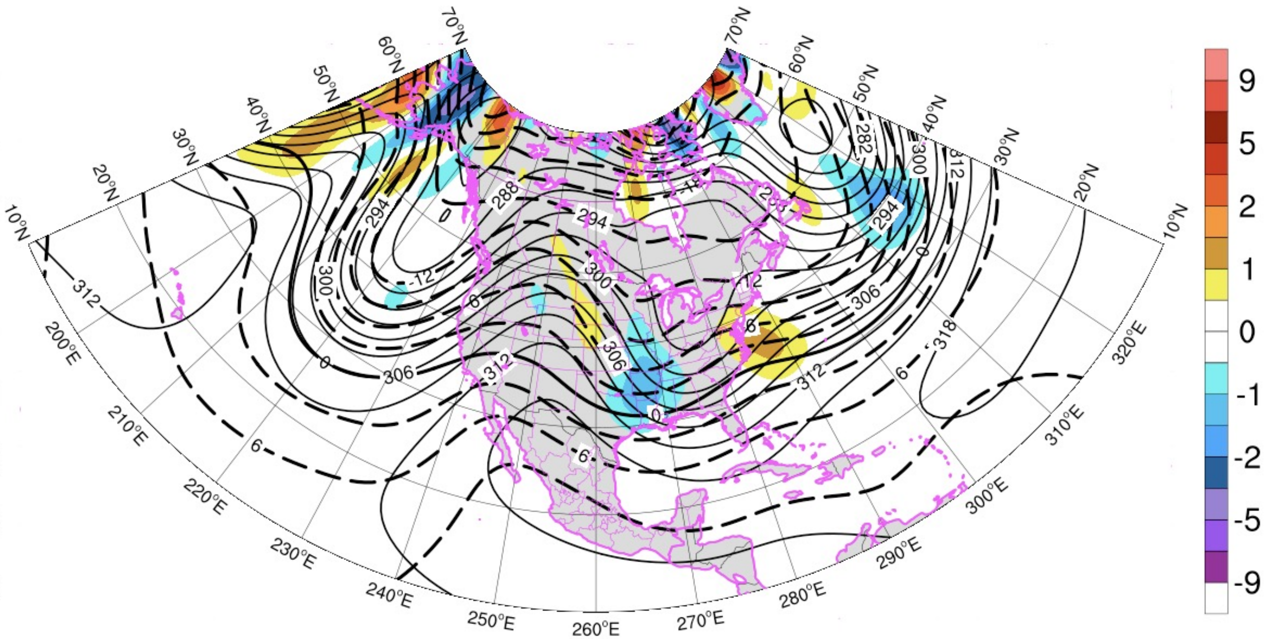
700 hPa Z and Total RHS QG Omega Forcing (traditional form) (smoothed)
0-h CMC Global forecast at 2021020100



- Precip in PA/NJ in region of QG ascent assoc. with **warm advection**
- Convection in NC to FL in region of QG ascent assoc. with **differential cyclonic vorticity advection**



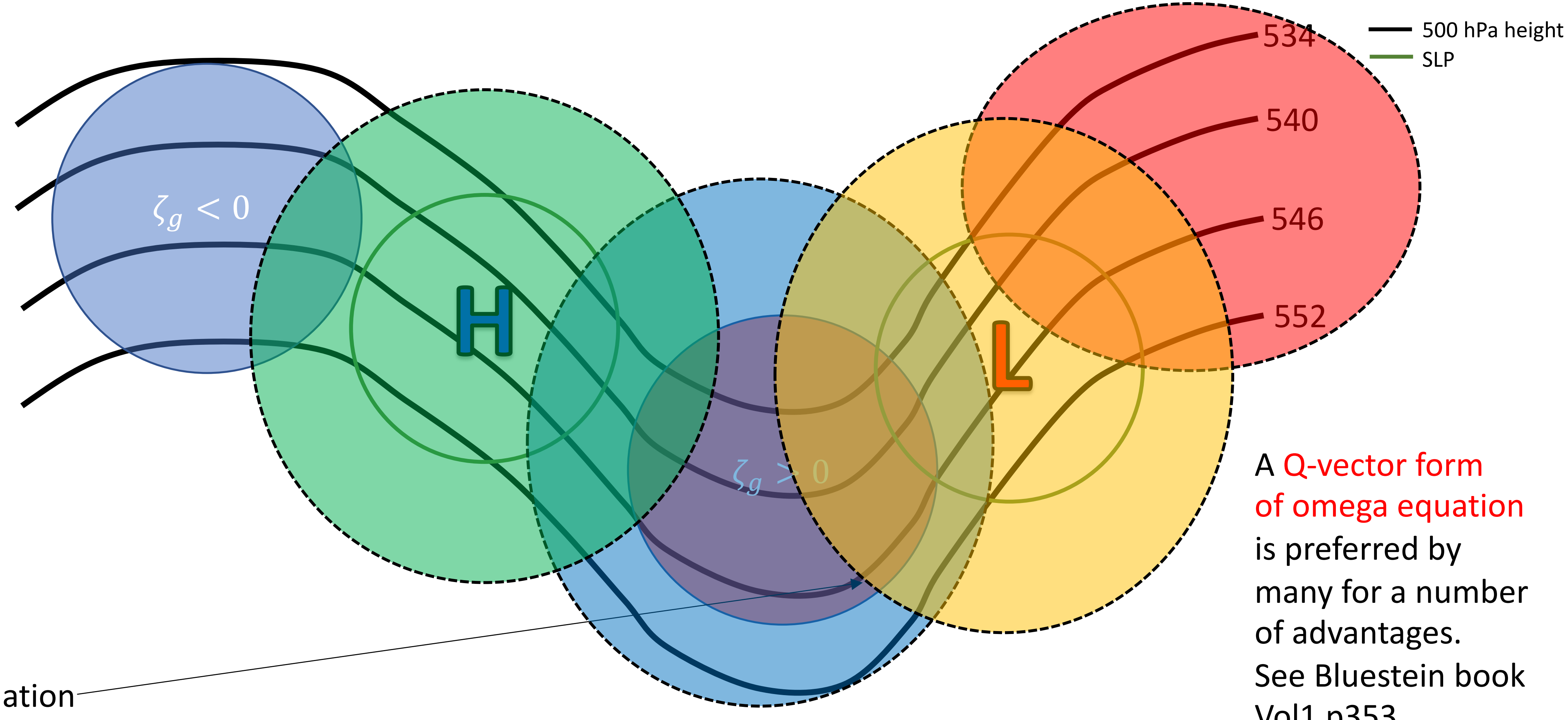
700 hPa Z and Differential Vorticity Advection Term B QG Omega Forcing (traditional form) (smoothed)
0-h CMC Global forecast at 2021020100



700 hPa Z, Temperature, and Thermal Advection Term C QG Omega Forcing (traditional form) (smoothed)
0-h CMC Global forecast at 2021020100

QG Omega Equation traditional form "problems"

- Forcing due to differential vorticity advection and thermal advection often do not coincide
- Terms are not Galilean invariant

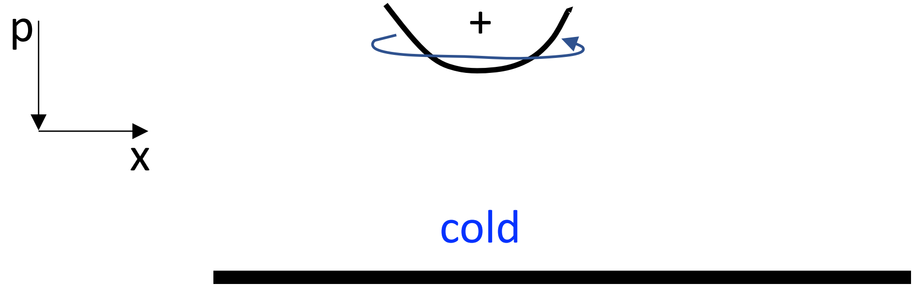


A Q-vector form of omega equation is preferred by many for a number of advantages. See Bluestein book Vol1 p353.

Geostrophic forcing and ageostrophic response

- Geostrophic advection acts to destroy thermal wind balance
- Ageostrophic circulation acts to restore/maintain thermal wind balance (TWB)

Thermal wind balance: $\frac{\partial \zeta_g}{\partial p} = -\frac{R_d}{f_0 p} \nabla_h^2 T$



$\frac{\partial \zeta_g}{\partial p} < 0 \rightarrow -\frac{R_d}{f_0 p} \nabla_h^2 T < 0$

$\therefore \nabla_h^2 T > 0$

$\therefore T_{min} \rightarrow$ cold

- Consider situation where differential cyclonic vorticity advection acts to make $\frac{\partial \zeta_g}{\partial p} \ll 0$; QG omega equation also says this is associated with rising motion
- TWB is destroyed → temperature is out of balance (not cold enough) with the increased strength of vorticity aloft (too strong for the temperature)
- How does rising motion (ageostrophic circulation) help to restore TWB?

Geostrophic forcing and ageostrophic response

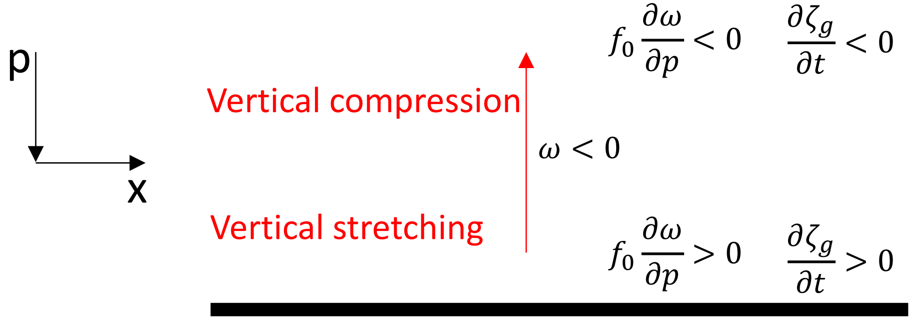
TWB: $\frac{\partial \zeta_g}{\partial p} = -\frac{R_d}{f_0 p} \nabla_h^2 T$

- Rising motion can restore TWB by:
  - Opposing changes to $\frac{\partial \zeta_g}{\partial p}$ (recall vorticity advection makes $\frac{\partial \zeta_g}{\partial p} \ll 0$)
  - Adjusting temperature to make RHS of TWB equation more negative

Geostrophic forcing and ageostrophic response

TWB: $\dfrac{\partial \zeta_g}{\partial p} = -\dfrac{R_d}{f_0 p}\nabla_h^2 T$

- Rising motion can restore TWB by:
  - Opposing changes to $\frac{\partial \zeta_g}{\partial p}$ (recall vorticity advection makes $\frac{\partial \zeta_g}{\partial p} \ll 0$)
  - Adjusting temperature to make RHS of TWB equation more negative
- Consider RHS of QG vorticity equation $\dfrac{d(\zeta_g + f)}{dt} = \boxed{f_0 \dfrac{\partial \omega}{\partial p}}$



$$\therefore \frac{\partial}{\partial t}\left(\frac{\partial \zeta_g}{\partial p}\right) > 0$$

Dynamic effect of $\omega < 0$ reduces effect of differential vorticity advection –

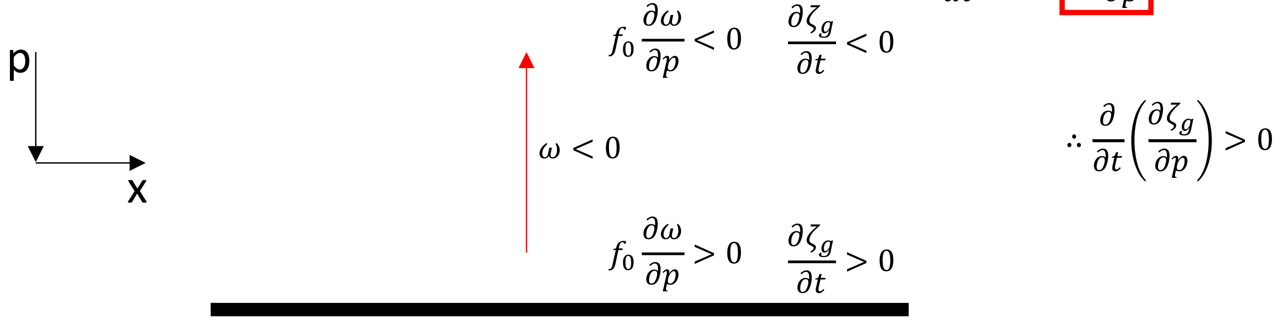
Decrease the change to the left hand side

Geostrophic forcing and ageostrophic response

TWB: $\frac{\partial \zeta_g}{\partial p} = -\frac{R_d}{f_0 p} \nabla_h^2 T$

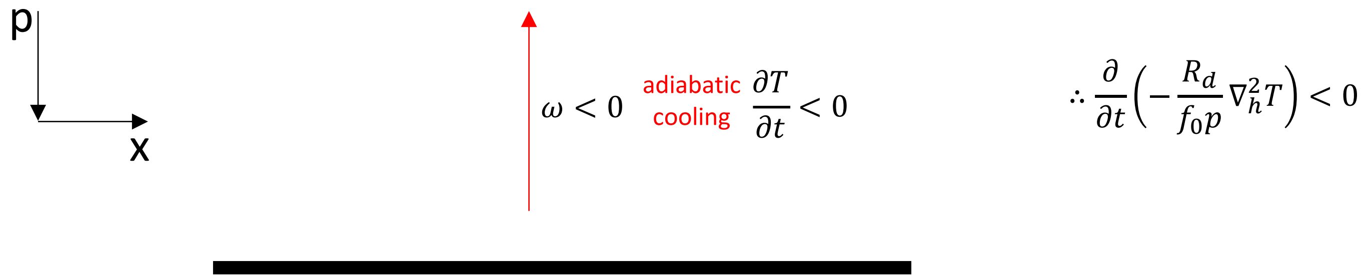
- Rising motion can restore TWB by:
  - Opposing changes to $\frac{\partial \zeta_g}{\partial p}$ (recall vorticity advection makes $\frac{\partial \zeta_g}{\partial p} \ll 0$)
  - Adjusting temperature to make RHS of TWB equation more negative

- Consider RHS of QG vorticity equation $\frac{d(\zeta_g + f)}{dt} = f_0 \frac{\partial \omega}{\partial p}$



Dynamic effect of $\omega < 0$ reduces effect of differential vorticity advection

- Consider RHS of QG thermodynamic equation $\frac{dT}{dt} = \frac{p}{R_d} \sigma \omega$



Thermodynamic effect of $\omega < 0$ is to adjust temperature field to be in TWB with new $\frac{\partial \zeta_g}{\partial p}$.

-- Increase the right hand side to match change to the left hand side