2D Boussinesq Equations

The equation of motion for 2D fluid with Boussinesq approximation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \qquad (1)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}+g\frac{\theta'}{\theta_0}, \qquad (2)$$

where variables, *u*, *w, p, ρ, θ,* are wind components in x and z directions, *p* is pressure, *ρ, θ,* is potential temperature. *ρ* is constant base state air density, and *θ* is constant base state potential temperature. $\theta' = \theta - \theta_0$ is the peturbation potential temperature. The second term on the right hand side of (2) is the buoyancy term.

$$\frac{\partial \theta'}{\partial t}+u\frac{\partial \theta'}{\partial x}+w\frac{\partial \theta'}{\partial z}=0 \qquad (3)$$

is the potential temperature conservation equation.

$\frac{\partial \text{Eq.(1)}}{\partial z}-\frac{\partial \text{Eq.(2)}}{\partial z}$ gives the y-vorticity equation:

$$\frac{\partial \eta}{\partial t}+u\frac{\partial \eta}{\partial x}+w\frac{\partial \eta}{\partial z}=-\frac{g}{\theta_0}\frac{\partial \theta'}{\partial x}, \qquad (4)$$

where

$$\eta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}.$$

For this 2D non-divergent flow, the velocity components can be written in term of stream function *ψ*,

$$u=\frac{\partial \psi}{\partial z}, w=-\frac{\partial \psi}{\partial x}, \qquad (5)$$

so that vorticity

$$\eta=\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial z^2}=\nabla^2\psi, \qquad (6)$$

or $\nabla^2\psi=\eta$, (7)

which is an elliptic equation which can be solved for ψ when η is known. In this term project, you are to use equations (3), (4) and (7) to construct a 2D model where prognostic state variables are η and θ' , and ψ is a diagostic state variable.

Eq. (3) and (4) are first integrated forward to obtain η and θ' at the future time level, then elliptic Eq.(7) is solved to obtain ψ . *u* and *w* are then obtained from (5). One time step is then complete.

Term Project Assignment – Part 1

Define a computational domain that is *H* deep, and *L* wide, and grid intervals *dx* and *dz*. Then $L = (nx - 1)\, dx$ and $H = (nz - 1)\, dz$ where *nx* and *nz* are the number of grid points defined from the left to right boundaries and from the bottom to top boundaries, respectively.

For the implementation lateral boundary conditions, it is convenient to define one extra column of grid points outside the left and right physical boundaries, so that the grid point index goes from -1 to *nx*+1. With Fortran, you can declare arrays like u(0:nx+1, nz) to match such index. With Python, all arrays have to start from 0 index.

For our term project, we will assume periodic lateral boundary conditions. For example, using Fortran, for *u*,

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u(0,:) = u(nx-1,:)
u(nx+1,:) = u(2,:)
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The above conditions should be set for all state variables. The top and bottom boundary conditions are rigid walls, therefore $w = 0$ at $z = 0$ and $z = H$.

Problem 1: Assume zero initial environmental flow, therefore initial stream function is zero everywhere. At the top and bottom boundaries, ψ remains zero always, which is the top and bottom boundary conditions for ψ .

For *H* and *L* = 10,000 m, and dx = dz = 100 m,

$$\eta = \eta_0 \cos^2\left(\frac{\pi}{2}\frac{r}{R}\right) \text{ for } r \le R \text{ otherwise } \eta = 0, \qquad (8)$$

where *r* is the radial distance from the domain center, and $R = \frac{1}{4} H$.

For the above vorticity distribution and $\eta_0 = 0.01\ \text{s}^{-1}$ solve for the stream function ψ using successive over-relaxation method. Confirm that your results are correct by plotting (within the physical domain without extra boundary points) the original η and η calculated from (6).

Calculate u and w from (5) and plot the wind vectors in the physical domain.

Problem 2: For constant flow u = 10 m/s, w = 0, and the initial vorticity given by (8), integration (4) without the right hand side buoyancy gradient term for 2000 s, and plot your η solutions at 500, 1000, 1500, 2000 s.

Use forward-in-time, upstream-in-space scheme, and Courant number 0.25, 0.5, 0.75, and 1.0 to do the integration. Briefly discuss the results.