A Semi-Lagrangian and Semi-Implicit Numerical Integration Scheme for the Primitive Meteorological Equations

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Abstract

A semi-Lagrangian algorithm is associated with the semi-implicit method in the integration of the shallow water equations on a rotating sphere. The resulting model is unconditionally stable and can be integrated with rather large time steps. Truncation errors remain reasonably small with time steps 25 times as large as those used with explicit integration schemes.

An analysis of the proposed method is performed and it indicates that the scheme is stable. Also, the results of a few integrations are presented and from these we conclude that the model is not very sensitive to the size of the time step provided that it does not exceed a value of the order of two or three hours.

1. Introduction

The explicit leapfrog time integration scheme is very inefficient when it is used in large scale atmospheric models. For a grid length of 200 km, it is generally found that a time step of the order 5 minutes has to be used. With a longer time step, there is a good chance that the gravity waves will become unstable and the predictions will be useless. Most explicit time integration schemes have to use rather small time steps. For a model that uses second order finite differences in both space and time, it was shown by Robert (1981) that the time truncation errors are 700 times smaller than the space truncation errors. This means that a much larger time step could be used without any significant loss of accuracy if it was possible to avoid the computational instability.

With a semi-implicit time integration scheme, one can use a time step of the order of 30 minutes on a 200 km grid. In an experiment performed by Robert, Henderson and Turnbull (1972) it was shown that the time truncation errors are of the order of 4 meters for the 500 mb geopotential for a five day forecast. This is still a rather small value and it indicates that even larger time steps could be used if a completely stable time integration scheme was made available.

It is possible to combine the semi-implicit scheme with a semi-Lagrangian scheme in order to obtain an algorithm that remains stable for very large time steps. It seems that this technique can be applied to the primitive meteorological equations. An experiment was carried out along these lines by Robert (1981) with a time step of two hours. A satisfactory 48 hr forecast was generated with a model of the shallow water equations. This model included a divergence diffusion term and a time filter. Later on, when these features were removed, some residual instability was observed. This instability was attributed to the fact that the Coriolis terms were not given a semi-implicit formulation, and also to the fact that the semi-Lagrangian technique was applied only to the vorticity equation.

A new formulation of the same model will be examined in the following sections of this report. An analysis of this formulation will be carried out in order to show that it is stable and an integration will be carried out without any diffusion terms and without any filters in order to demonstrate that it works reasonably well. The sensitivity to the size of the time step will also be examined.

2. Formulation of the model

In a conformal projection of a rotating sphere, the shallow water equations take the following form:

$$\frac{dU}{dt}-f\overline{V}^{t}+\frac{\overline{\partial\phi}^{t}}{\partial X}+K\frac{\partial S}{\partial X}=0 \quad (1)$$

$$\frac{dV}{dt}+f\overline{V}^{t}+\frac{\overline{\partial\phi}^{t}}{\partial Y}+K\frac{\partial S}{\partial Y}=0 \quad (2)$$

$$\frac{d\phi_T}{dt}+\phi_0\bar{D}^t+(\phi_T-\phi_0)D=0 \quad (3)$$

This formulation includes the time discretization which will be defined below but first, we will define the various quantities

$$K=\frac{1}{2}(U^2+V^2) \quad (4)$$

$$D=S\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right) \quad (5)$$

$$f=2\Omega\sin\varphi \quad (6)$$

$$S=m^2 \quad (7)$$

$$U=\frac{u}{m} \quad (8)$$

$$V=\frac{v}{m} \quad (9)$$

$$m=\frac{\Delta X}{\Delta\chi} \quad (10)$$

$$\phi_T=\phi-\phi_G \quad (11)$$

In these equations, Ω is the angular velocity of rotation of the earth, φ is the latitude, $\Delta\chi$ is a distance on earth and ΔX is the corresponding distance in the projection, μ and v are the real wind components, ϕ is the geopotential of the free surface and ϕ_G is the geopotential of the ground at the bottom of the fluid so that ϕ_T is the thickness of the fluid layer and ϕ_0 is a constant roughly equal to the mean value of ϕ_T.

In order to carry out the calculations, three points P_1, P_2 and P_3 are defined with the following coordinates

$$P_1(X,\ Y,\ t+\Delta t) \quad (12)$$

$$P_2(X-a,\ Y-b,\ t) \quad (13)$$

$$P_3(X-2a,\ Y-2b,\ t-\Delta t) \quad (14)$$

Here it is obvious that P_2 is the mid point of the interval from P_1 to P_3 and a and b are defined as follows

$$a=\Delta tS(P_2)U(P_2) \quad (15)$$

$$b=\Delta tS(P_2)V(P_2) \quad (16)$$

Also, the time derivatives and the time averages are evaluated as follows

$$\frac{dF}{dt}=\frac{F(P_1)-F(P_3)}{2\Delta t} \quad (17)$$

$$\bar{F}^t=\frac{F(P_1)+F(P_3)}{2} \quad (18)$$

and all other terms in the above equations are evaluated at point P_2.

In plain language, we are taking a trajectory over a time interval of $2\Delta t$. This trajectory terminates at a grid point. The end points of the trajectory are used to compute the time derivatives and the time averages. All other quantities are computed at the mid point of the trajectory. This means that the model uses centered differences of second order accuracy.

3. Stability analysis

An attempt will be made to find solutions of eqs. (1), (2) and (3) in terms of the exponential

$$E=e^{i(\omega t+kx+ly)} \quad (19)$$

For this function we have

$$\frac{dE}{dt}=\frac{1}{2\Delta t}[E(P_1)-E(P_3)] \quad (20)$$

$$\frac{dE}{dt}=\frac{1}{2\Delta t}[e^{i\omega\Delta t}-e^{-i(\omega\Delta t+2ka+2lb)}]E \quad (21)$$

$$\frac{dE}{dt}=\frac{2}{2\Delta t}[e^{i(\omega\Delta t+ka+lb)}-e^{-i(\omega\Delta t+ka+lb)}]E(P_2) \quad (22)$$

$$\frac{dE}{dt}=\frac{i}{\Delta t}E(P_2)\sin(\omega\Delta t+ka+lb) \quad (23)$$

and in a similar fashion we also have

$$\bar{E}^t=E(P_2)\cos(\omega\Delta t+ka+lb) \quad (24)$$

For simplicity we will use

$$f=2\Omega \quad (25)$$

$$m=1 \quad (26)$$

and the solutions will be given the form of a weak perturbation superimposed on a steady basic state

$$U=U^*+U'E \quad (27)$$

$$V=V'E \quad (28)$$

$$\phi=\phi^*+\phi'E \quad (29)$$

$$\phi_T=\phi_0+\phi'E \quad (30)$$

where U^* is a constant and

$$\frac{\partial\phi^*}{\partial Y}=-2\Omega U^* \quad (31)$$

It should be noted that for this particular case, the underlying topography

$$\phi_G=\phi-\phi_T=\phi^*-\phi_0 \quad (32)$$

has the same slope along the Y-axis as the free surface.

Substitution in the shallow water equations gives the following result after dividing by $E(P_2)$.

$$\frac{i}{\Delta t}U's-2\Omega V'c+ik\phi'c=0 \tag{33}$$

$$\frac{i}{\Delta t}V's+2\Omega U'c+il\phi'c=0 \tag{34}$$

$$\frac{i}{\Delta t}\phi's+i\phi_0(kU'+lV')c=0 \tag{35}$$

where

$$s=\sin(\omega\Delta t+ka+lb)=\sin[\Delta t(\omega+kU^*)] \tag{36}$$

$$c=\cos(\omega\Delta t+ka+lb)=\cos[\Delta t(\omega+kU^*)] \tag{37}$$

Elimination of the unknown amplitudes U', V', and ϕ' in eqs. (33), (34) and (35) gives the frequency equations.

$$s\{s^2-c^2[4\Omega^2\Delta t^2+\phi_0(k^2+l^2)\Delta t^2]\}=0 \tag{38}$$

and from this equation we obtain the following frequencies

$$\omega=-kU^* \tag{39}$$

$$\omega=-kU^*\pm\frac{1}{\Delta t}\tan^{-1}[4\Omega^2\Delta t^2+\phi_0(k^2+l^2)\Delta t^2]^{1/2} \tag{40}$$

It is quite clear from this result that the solutions are always stable.

4. Integration of the model

The model described in section 2 will be integrated on a 61 by 61 grid in a polar stereographic projection. The grid covers North America and part of the adjacent oceans. The grid length is 190.5 km at 60°N. Fourth order differences are used to compute all space derivatives and fourth order interpolation is used to compute values along the trajectories.

At the boundaries, the cross flow is maintained at its initial value, the gradient of the geopotential normal to the boundary is maintained in geostrophic equilibrium and the tangential wind component is maintained at the value computed at the nearest interior point. At the first set of interior points, a strong diffusion term is used in the three equations in order to eliminate the noise generated at the boundaries.

This is the only filtering used in the model. Over the interior part of the grid, there are no explicit diffusion terms and no filters. On the other hand, it must be noted that the repeated interpolations required by the semi-Lagrangian method have a filtering effect on the variables. On such a fine grid and with fourth order interpolation, the amount of filtering is negligible.

Initialization is performed by using non diver-

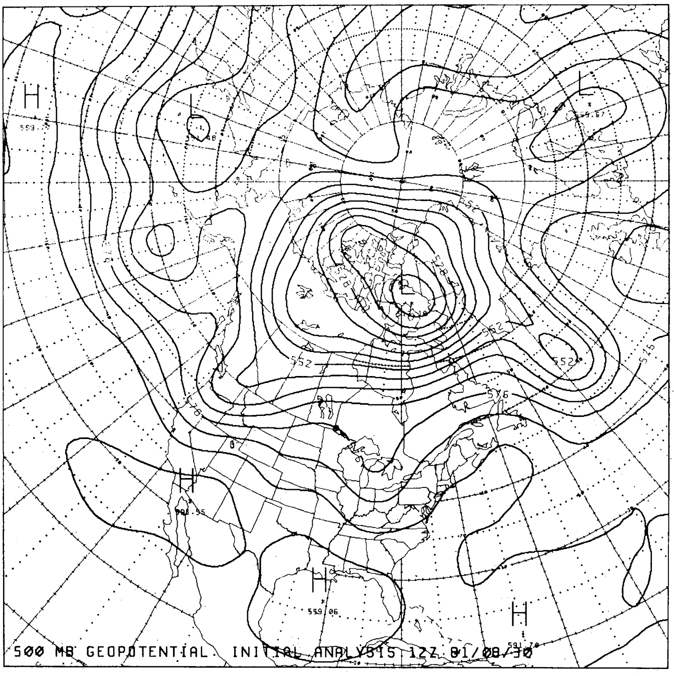

Fig. 1 Initialized 500 mb geopotential at 12:00 GMT 30 August 1981.

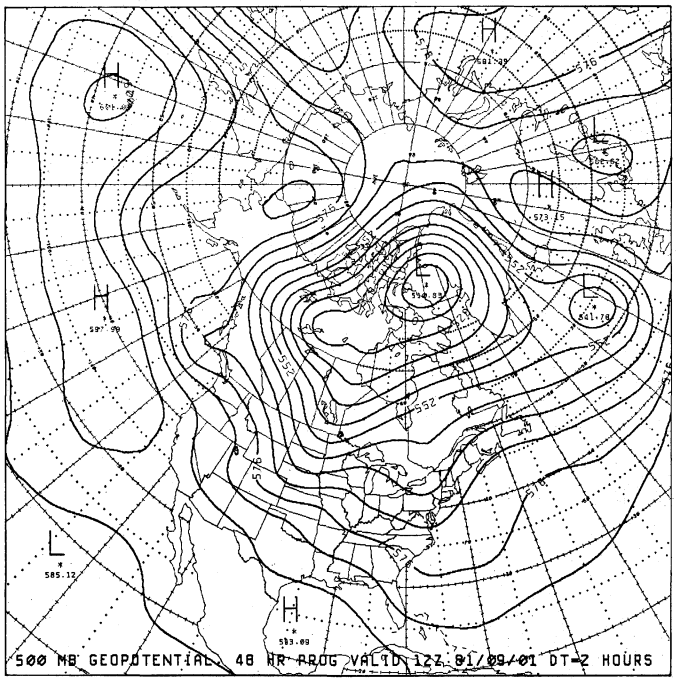

Fig. 2 48-hour forecast of the 500 mb geopotential valid at 12:00 GMT 1 September 1981. Semi-Lagrangian and semi-implicit model with a two hour time step.

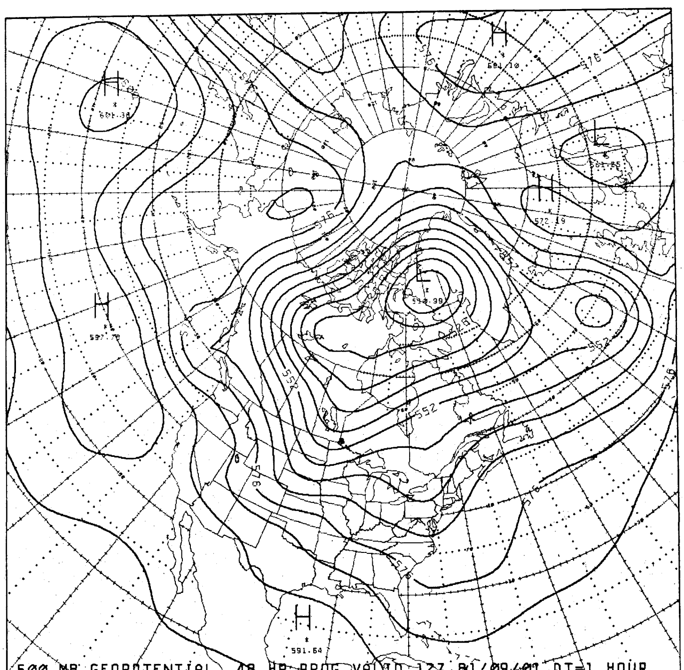

Fig. 3 48-hour forecast of the 500 mb geopotential valid at 12:00 GMT 1 September 1981. Semi-Lagrangian and semi-implicit model with a one hour time step.

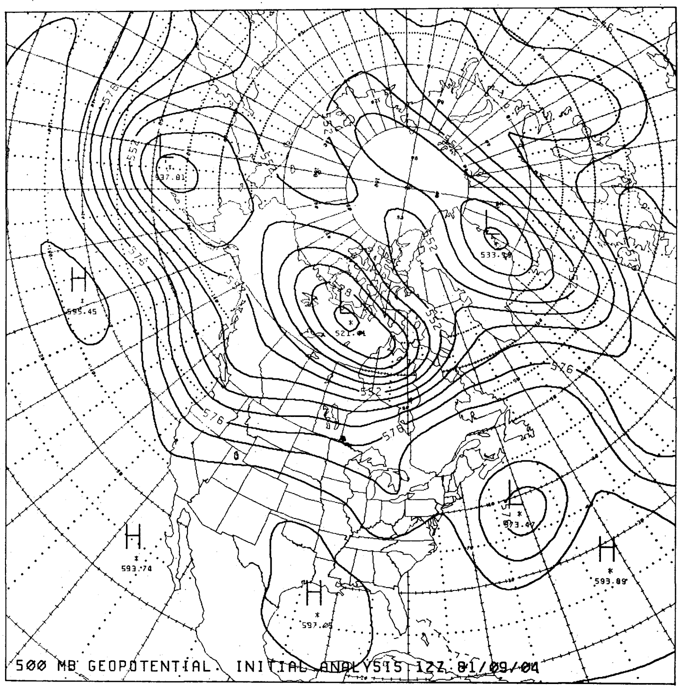

Fig. 4 Same as Fig. 1 for 12:00 GMT 4 Septmeber 1981.

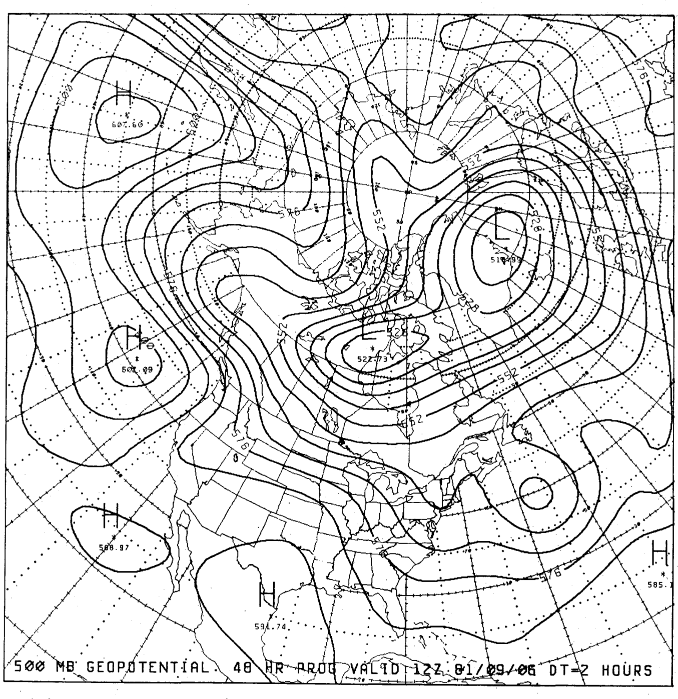

Fig. 5 Same as Fig. 2 for 12:00 GMT 6 September 1981.

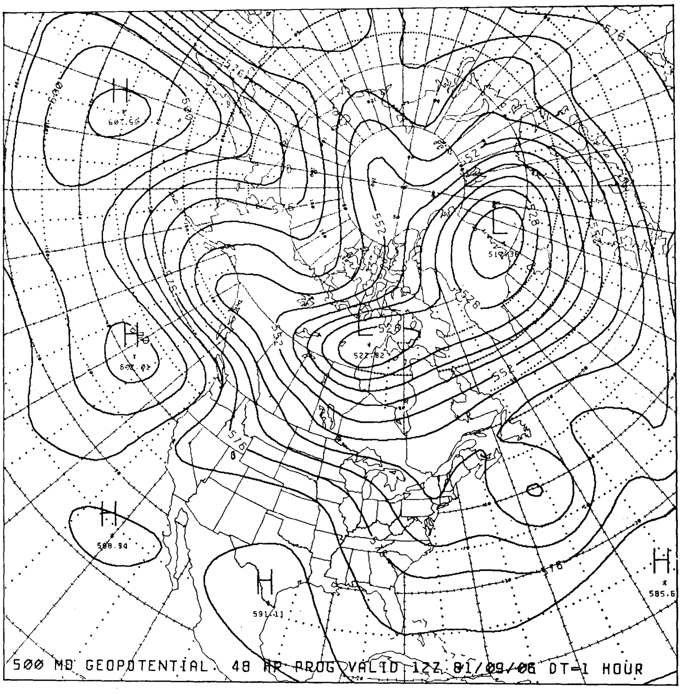

Fig. 6 Same as Fig. 3 for 12:00 GMT 6 September 1981.

gent winds to start the integration and by using a geopotential obtained from the balance equation. Topography has not been included in this model.

A first integration is performed from the 500 mb analysed variables at 12:00 GMT 30 August 1981. The initialized 500 mb geopotential is presented in Fig. 1. The 500 mb geopotential after 48 hours produced from an integration of the semi-Lagrangian and semi-implicit model is given in Fig. 2. A time step of two hours was used for this integration. The result obtained with a time step of one hour is also shown in Fig. 3. The rather small differences between these integrations indicate that the truncation errors associated with this integration scheme are small even with a time step as large as two hours. We can also conclude that the scheme is computationally stable.

A second integration is also carried out from the 500 mb analysed variables at 12:00 GMT 4 September 1981. The initialized 500 mb geopotential is given in Fig. 4 followed by the 48 hour prediction with a time step of two hours presented in Fig. 5. Finally, the same prediction with a time step of one hour is shown in Fig. 6. Here again, we note that there are only small differences between the two predictions.

5. Conclusion

A semi-Lagrangian time integration scheme can easily be combined with the semi-implicit algorithm. This combination enables us to increase the time step by another factor of four to six. The resulting predictions are essentially identical to those generated with much shorter time steps. Semi-Lagrangian advection involves more calculations than the regular Eulerian advection but the increase in the number of computations per time step is more than compensated by the economy arising from the use of a very large time step.

The true test of the proposed integration scheme will consist in trying to use it in a complete multi-level atmospheric model. Such a test is currently in preparation. The results will be published as soon as they become available.

References

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プリミティブ方程式のためのセミラグランジュ法とセミインプリシット法を併用するスキーム

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回転系における浅水方程式の時間積分にセミインプリシット法とセミラグランジュ法を併用したスキームを用いる。このスキームは数値計算上絶対安定であり，かなり大きなタイムステップを用いて積分することが可能となる。時間積分による打切り誤差は，エクスプリシット法の時の 25 倍のタイムステップを用いても充分に小さかった。

線型安定解析によるとこのスキームは安定である。また，このスキームを用いた数例の時間積分の結果によると，タイムステップが 2～3 時間のオーダーを越さない限り，結果はタイムステップの大きさに依存しない。