Mesoscale Meteorological Modeling

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When $\alpha^2 > 4$, $\cos\omega_r \Delta t = 0$, so that the imaginary component can be written as

$$\lambda = -\tfrac{1}{2}\alpha \pm \tfrac{1}{2}\sqrt{\alpha^2 - 4}.$$

To ascertain whether this quantity is less than or greater than unity, let

$$\alpha = 2 + \varepsilon,$$

where $\varepsilon > 0$, so that

$$\lambda = -1 - \tfrac{1}{2}\varepsilon \pm \tfrac{1}{2}\sqrt{4\varepsilon + \varepsilon^2}$$

Since either root is possible

$$\lambda = -1 - \tfrac{1}{2}\varepsilon - \tfrac{1}{2}\sqrt{4\varepsilon + \varepsilon^2} < -1,$$

($|\lambda| > 1$) so that when $\alpha^2 > 4$, the leapfrog scheme is linearly unstable. Since $\alpha^2 = 4C^2 \sin^2 k\,\Delta x$, stability is retained only when

$$C^2 \sin^2 k\,\Delta x \leq 1,$$

or since the maximum value of $\sin^2 k\,\Delta x$ is unity for a $4\,\Delta x$ wave ($k = \pi/2$), then

$$|C| \leq 1$$

is a necessary and sufficient condition for the linear stability of the scheme.

The ratio of the predicted phase speed to the advecting velocity for this technique can be obtained by dividing the imaginary by the real component for $\alpha^2 \leq 4$ and solving for the phase speed. Since $|\lambda| = 1$, however, it is also possible to use either the imaginary or real components separately to obtain the phase speed. Using the imaginary part, therefore, gives the ratio of the calculated to analytic phase speeds as

$$\frac{\tilde{c}_\phi}{U} = \frac{1}{Uk\,\Delta t}\sin^{-1}\left(\pm\frac{\alpha}{2}\right).$$

Because of the quadratic form of (10-18), two wave solutions occur. One moves downstream ($\tilde{c}_\phi > 0$ when $U > 0$) and is related to the real solution of the advection equation, and the other travels upstream and is called the *computational mode*. The computational mode occurs because the leapfrog is a second-order difference equation. Such separation of solutions by the centered-in-time, leap-frog scheme can be controlled by occasionally averaging in time to assure that the even and odd time steps remain consistent with one another. As long as the time steps are consistent the amplitude of the computational mode is small.

Values of λ and $\tilde{c}_\phi$ for different values of C and wavelength are displayed in Table 10-1. Although the leap-frog scheme preserves amplitudes exactly as long as $|C| \leq 1$, the accuracy of the phase representation deteriorates markedly for the shorter wavelengths. Because the numerical representation of these waves travels more slowly than the true solution, the scheme is said to be *dispersive* since when waves of different wavelengths are linearly superimposed, they will travel with different speeds relative to one another even if the advecting velocity is a constant. The retention of these dispersive shorter waves in the solution can cause computational problems through nonlinear instability, as discussed in Section 10.6.[6] The important conclusion obtained from the analysis of the leap-frog scheme is that *the exact representation of the amplitude does not by itself guarantee successful simulations since the fictitious dispersion of waves of different lengths can generate errors.* Baer and Simons (1970), for example, have reported that in approximating nonlinear advection terms, individual energy components may have large errors when the total energy has essentially none. They further conclude that neither conservation of integral properties nor satisfactory prediction of amplitude is sufficient to justify confidence in the results—one must also assure the accurate calculation of phase speed.

In both the forward-upstream and leap-frog schemes that we have examined, the time step must be less than or equal to the time it takes for changes at one grid point to be translated by advection to the next grid point downstream. When we generalize this result to all types of wave propagation, the need to filter rapidly moving waves, which are not considered important on the mesoscale, is apparent. This is the reason that scale analysis is used to derive simplified conservation relations [e.g., the anelastic conservation of mass equation, (3-11)] so that sound waves can be eliminated as a possible solution, as shown in Section 5.2.2.

10.1.2 Subgrid Scale Flux

As shown by (7-7), the subgrid scale correlation terms can be represented as the product of an exchange coefficient and the gradient of the appropriate dependent variable. This relation can be written, for example, as

$$\frac{\partial\bar{\phi}}{\partial t} = \frac{\partial}{\partial z} K \frac{\partial\bar{\phi}}{\partial z} \simeq \frac{\phi_i^{\tau+1} - \phi_i^{\tau}}{\Delta t} = K_{i+\frac{1}{2}} \frac{\phi_{i+1}^{\tau} - \phi_i^{\tau}}{(\Delta z)^2} - K_{i-\frac{1}{2}} \frac{\phi_i^{\tau} - \phi_{i-1}^{\tau}}{(\Delta z)^2}, \quad (10\text{-}19)$$

where $\Delta z = z(i+1) - z(i) = z(i) - z(i-1)$ and ϕ represents any one of the dependent variables. This equation is often referred to as the *diffusion equation.* To study the linear stability of this scheme, the exchange coefficient is assumed a constant ($K_{i+\frac{1}{2}} = K_{i-\frac{1}{2}} = K$), and (10-19) is written as

$$\phi_i^{\tau+1} = \phi_i^{\tau} + K\frac{\Delta t}{(\Delta z)^2}(\phi_{i+1}^{\tau} - 2\phi_i^{\tau} + \phi_{i-1}^{\tau}). \quad (10\text{-}20)$$

The exact solution to the diffusion equation (the left-hand side of (10-19) with K equal to a constant, i.e., $\partial\bar{\phi}/\partial t = K\,\partial^2\bar{\phi}/\partial z^2$) can be determined by assuming

$$\bar{\phi} = \phi_0 e^{i(kz+\omega t)} = \phi_0 e^{-\omega_i t} e^{i(k_r z + \omega_r t)},$$

where no damping in the z direction is permitted (i.e., $k_i \equiv 0$). Substituting this expression into the linearized diffusion equation and simplifying, yields

$$i\omega_r - \omega_i = -Kk^2$$

where the subscript r on k has been eliminated to simplify the notation. Equating real and imaginary components shows that $\omega_r \equiv 0$ so that the exact solution can be written as

$$\bar{\phi} = \phi_0 e^{-Kk^2 t} e^{ikz}.$$

Expressing the dependent variables as a function of frequency and wavenumber, (10-20) can be rewritten as

$$\psi^1 = 1 + \gamma(\psi_1 - 2 + \psi_{-1}) = 1 + 2\gamma(\cos k\,\Delta z - 1),$$

where $\gamma = K\,\Delta t/(\Delta z)^2$ and $\psi_1 + \psi_{-1} = 2\cos k\,\Delta z$. The nondimensional parameter γ is called the *Fourier number*. Equating real and imaginary components yields

$$\lambda \cos \omega_r\,\Delta t = 1 + 2\gamma(\cos k\,\Delta z - 1),$$
$$\lambda \sin \omega_r\,\Delta t = 0.$$

Since $\sin \omega_r\,\Delta t$ must be identically equal to zero, $\omega_r\,\Delta t$ and, therefore, the phase speed are also equal to zero. Thus the solution to (10-20) does not propagate as a wave but amplifies or decays in place. Since $\cos \omega_r\,\Delta t = 1$, the real part can be divided by the analytic solution,[7] $\lambda_a = e^{-Kk^2\,\Delta t} = e^{-\gamma(2\pi)^2/n^2}$ and rewritten as

$$\frac{\lambda}{\lambda_a} = \frac{1 + 2\gamma(\cos k\,\Delta z - 1)}{e^{-\gamma(2\pi)^2/n^2}},$$

where n is the number of grid points per wavelength. For very long waves ($n \to \infty$) $\lambda_a = 1$ and $\lambda = 1$ since $\cos k\,\Delta z = \cos(2\pi/2)\,\Delta z = 1$, and, therefore, no damping or amplification occurs. For the shortest waves that can be resolved ($L = 2\,\Delta z$; $n = 2$),

$$\lambda = 1 - 4\gamma.$$

To assure that the magnitude of λ is less than unity and, therefore, computationally stable, 4γ must be less than or equal to 2 or

$$\gamma \le \tfrac{1}{2}.$$

The condition $\gamma = \frac{1}{2}$, however, causes λ to switch between $+1$ and -1 each

application of (10-20), but the analytic solution is $\lambda_a = e^{-9.9} \doteq 0.00005$. This unrealistic response of $2\,\Delta z$ wavelength features can cause computational problems in a nonlinear model as is discussed in Section 10-6. To eliminate $2\,\Delta z$ waves at each application of (10-20), λ can be set to zero for a $2\,\Delta z$ wave resulting in $\lambda = \frac{1}{4}$. Thus the standard requirement specified in using this scheme is that

$$\gamma = K\,\Delta t/(\Delta x)^2 \le \tfrac{1}{4},$$

with the expectation that γ is close to $\frac{1}{4}$ so that the presence of $2\,\Delta z$ waves is minimized.

Up to this point, the approximation to the advective and subgrid scale flux terms have always been defined at the current time step (i.e., ϕ_i^τ). The predicted dependent variable $\phi_i^{\tau+1}$ only enters through the time tendency term. Such schemes are referred to as *explicit* and can be written in general as

$$\phi^{\tau+1} = f(\underset{\sim}{\phi}^\tau),$$

where the function f, can include spatial derivatives of ϕ^τ as well as the variable itself. The tilde under ϕ indicates that $\phi^{\tau+1}$ at a specific point can be dependent on values of ϕ^τ at other grid points.

In contrast, an *implicit* scheme uses information from the future time step, as well as present values. For this case

$$\phi^{\tau+1} = f(\underset{\sim}{\phi}^{\tau+1}, \underset{\sim}{\phi}^\tau).$$

In general the use of an implicit representation permits longer time steps than the explicit form without causing linear instability. An implicit form of the left-hand equation in (10-19) for variable Δz can be written (e.g., Paegle *et al.*, 1976) as

$$\frac{\phi^{\tau+1} - \phi^\tau}{\Delta t} = \frac{1}{\Delta z_j}\left[K_{j+\frac{1}{2}}\frac{\beta_\tau(\phi_{j+1}^\tau - \phi_j^\tau) + \beta_{\tau+1}(\phi_{j+1}^{\tau+1} - \phi_j^{\tau+1})}{\Delta z_{j+\frac{1}{2}}} - K_{j-\frac{1}{2}}\frac{\beta_\tau(\phi_j^\tau - \phi_{j-1}^\tau) + \beta_{\tau+1}(\phi_j^{\tau+1} - \phi_{j-1}^{\tau+1})}{\Delta z_{j-\frac{1}{2}}}\right] \quad (10\text{-}21)$$

where $\beta_\tau + \beta_{\tau+1} = 1$, $\Delta z_j = z_{j+\frac{1}{2}} - z_{j-\frac{1}{2}}$, $\Delta z_{j+1} = z_{j+1} - z_j$, and $\Delta z_{j-1} = z_j - z_{j-1}$. The use of β_τ and $\beta_{\tau+1}$ weights the current and future contributions to the numerical approximation of the left-hand side of (10-19). Note that when $\beta_{\tau+1} = 0$ and $\Delta z_j = \Delta z_{j+1} = \Delta z_{j-1} = \Delta z$, the scheme reverts back to the explicit scheme given by the right side of (10-19). Linearizing (10-21) by setting $K_{j+\frac{1}{2}}$ and $K_{j-\frac{1}{2}}$ equal to a constant, using a constant grid interval Δz, and representing the dependent variable in terms of wavenumber and frequency results in

$$\psi^1 = 1 + \gamma[\beta_\tau(\psi_1 - 2 + \psi_{-1}) + \beta_{\tau+1}(\psi_1^1 - 2\psi^1 + \psi_{-1}^1)],$$

TABLE 10-2

Values of the Ratio of the Computational to Analytic Damping as a Function of Wavelength for Different Forms of the Forward-in-time, Centered-in-space Approximation to the Linearized Diffusion Equation ($\partial\phi/\partial t = K\,\partial^2\phi/\partial z^2$)[a]

Scheme	Wavelength	γ = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Forward-in-time centered-in-space diffusion											
Explicit	$2\Delta z$	1.610	1.440	−3.863	−31.094	< −100					
	$4\Delta z$	1.024	0.983	0.839	0.537	0.0					
$\beta_\tau = 1$	$10\Delta z$	1.001	0.999	0.997	0.992	0.986	$\lambda > 1$ for a $2\Delta x$ wave				
	$20\Delta z$	1.000	1.000	1.000	1.000	0.999					
Implicit	$2\Delta z$	1.725	2.554	2.272	−4.202	−34.761	< −100	< −100	< −100	< −100	< −100
	$4\Delta z$	1.038	1.053	1.030	0.952	0.792	0.517	0.079	−0.584	−1.555	−2.948
$\beta_\tau = 0.7$	$10\Delta z$	1.001	1.001	1.001	1.000	0.998	0.996	0.992	0.988	0.982	0.975
	$20\Delta z$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999	0.999
	$2\Delta z$	1.789	3.085	4.829	5.758	0.00	−33.91	< −100	< −100	< −100	< −100
	$4\Delta z$	1.047	1.092	1.129	1.150	1.145	1.099	0.993	0.800	0.485	0.00
$\beta_\tau = 0.5$	$10\Delta z$	1.001	1.003	1.004	1.005	1.006	1.007	1.007	1.008	1.008	1.008
	$20\Delta z$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.001	1.001	1.001
	$2\Delta z$	1.845	3.507	6.718	12.711	23.17	38.97	54.11	33.16	< 100	< 100
	$4\Delta z$	1.055	1.126	1.211	1.307	1.414	1.529	1.648	1.766	1.875	1.965
$\beta_\tau = 0.3$	$10\Delta z$	1.002	1.004	1.006	1.009	1.013	1.017	1.021	1.026	1.031	1.037
	$20\Delta z$	1.000	1.000	1.000	1.001	1.001	1.001	1.001	1.002	1.002	1.003
	$2\Delta z$	1.916	3.999	8.779	19.93	46.35	> 100	> 100	> 100	> 100	> 100
	$4\Delta z$	1.067	1.170	1.310	1.491	1.717	1.998	2.344	2.769	3.290	3.931
$\beta_\tau = 0.1$	$10\Delta z$	1.002	1.005	1.010	1.016	1.023	1.031	1.040	1.050	1.062	1.074
	$20\Delta z$	1.000	1.000	1.001	1.001	1.002	1.002	1.003	1.004	1.004	1.005

[a] Computed by C. Martin.

where, as with the explicit scheme, $\gamma = K\,\Delta t/(\Delta z)^2$. Since

$$\psi_1^1 = \psi^1\psi_1, \qquad \text{and} \qquad \psi_{-1}^1 = \psi^1\psi_{-1},$$
$$\psi^1 = 1 + \gamma\beta_\tau(\psi_1 - 2 + \psi_{-1}) + \gamma\beta_{\tau+1}\psi^1(\psi_1 - 2 + \psi_{-1}),$$

or

$$\psi^1 = \frac{[1 + \gamma\beta_\tau(\psi_1 + \psi_{-1} - 2)]}{[1 - \gamma\beta_{\tau+1}(\psi_1 + \psi_{-1} - 2)]} = \frac{1 + 2\gamma\beta_\tau(\cos k\,\Delta z - 1)}{1 - 2\gamma\beta_{\tau+1}(\cos k\,\Delta z - 1)} = \lambda,$$

where as with the analysis of the explicit representation, the imaginary part is zero so that $\lambda = \psi^1$.

Values of the ratio of the computational approximation of the damping to the analytic damping λ/λ_a are presented in Table 10-2 as a function of wavelength and β_τ. For a given value of γ, the $2\Delta z$ wave is most poorly represented. In addition, the $2\Delta z$ wave is always insufficiently damped and often the value of λ is negative yielding a wave whose amplitude reverses (flip flops) each time step. The solutions become more accurate as γ becomes smaller, and the implicit representation gives reasonable results for large wavelengths even when the explicit form is linearly unstable for all spatial scales.

Equation (10-21) can be written in the following form

$$-\frac{\Delta t K_{j-\frac{1}{2}}\beta_{\tau+1}}{\Delta z_j \Delta z_{j-\frac{1}{2}}}\phi_{j-1}^{\tau+1} + \left[1 + \frac{\Delta t K_{j+\frac{1}{2}}\beta_{\tau+1}}{\Delta z_j \Delta z_{j+\frac{1}{2}}} + \frac{\Delta t K_{j-\frac{1}{2}}\beta_{\tau+1}}{\Delta z_j \Delta z_{j-\frac{1}{2}}}\right]\phi_j^{\tau+1}$$
$$-\frac{\Delta t K_{j+\frac{1}{2}}\beta_{\tau+1}}{\Delta z_j \Delta z_{j+\frac{1}{2}}}\phi_{j+1}^{\tau+1}$$
$$= \phi_j^\tau + \frac{\Delta t}{\Delta z_j}\left[\frac{K_{j+\frac{1}{2}}\beta_\tau(\phi_{j+1}^\tau - \phi_j^\tau)}{\Delta z_{j+\frac{1}{2}}} - \frac{K_{j-\frac{1}{2}}\beta_\tau(\phi_j^\tau - \phi_{j-1}^\tau)}{\Delta z_{j-\frac{1}{2}}}\right] \qquad (10\text{-}21a)$$

and solved for nonperiodic boundary conditions using a procedure described in Section 10.2. Its solution for periodic boundary conditions is given in Appendix A.

When $\beta_\tau = \beta_{\tau+1}$ this representation is called the Crank–Nicholson scheme. Paegle *et al.* (1976) have presented results that show that $\beta_{\tau+1} = 0.75$ provides a representation as accurate as the explicit scheme but with a much longer permissible time step. Figure 10-3, reproduced from Mahrer and Pielke (1978b), illustrates predictions of the growth of a heated boundary layer using both the explicit representation of diffusion given by (10-19) and the implicit form (10-21) with $\beta_{\tau+1} = 0.75$. As reported in that paper, use of the implicit form permitted a much longer time step so that that calculation ran 17 times faster than when the explicit form was used.

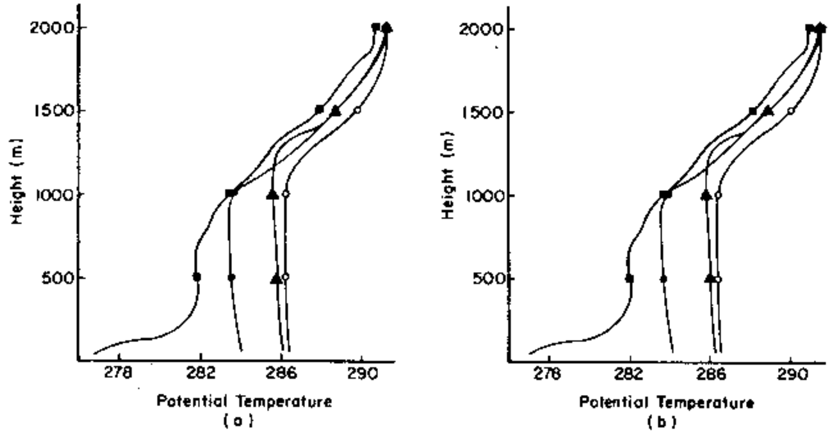

Fig. 10-3. Vertical profiles of the potential temperature for Wangara Day 33 with (a) the implicit scheme ($\beta_{\tau+1} = 0.75$) and (b) the explicit scheme ($\beta_{\tau+1} = 0$) (from Mahrer and Pielke, 1978b), where -■-, 0900; -●-, 1200; -▲-, 1500; -○-, 1700.

10.1.3 Coriolis Terms

The implicit scheme can also be shown to be a necessity for the Coriolis terms. The terms dealing with the rotation of the earth [see (4-20)] are already in linear form and with $L_z \ll L_x$ can be written as

$$\partial \bar{u}/\partial t = f\bar{v}; \qquad \partial \bar{v}/\partial t = -f\bar{u} \tag{10-22}$$

If these relations are approximated using an explicit representation, they are written as

$$(u_i^{\tau+1} - u_i^{\tau})/\Delta t = f v_i^{\tau}; \qquad (v_i^{\tau+1} - v_i^{\tau})/\Delta t = -f u_i^{\tau}. \tag{10-23}$$

Rewriting the dependent variables in terms of frequency and wavenumber, and rearranging yields

$$\hat{u}(\psi^1 - 1) - \hat{v}\,\Delta t f = 0,$$
$$\hat{u} f\,\Delta t + \hat{v}(\psi^1 - 1) = 0,$$

where $\hat{u}$ and $\hat{v}$ are functions of ω and k. In matrix form these equations can be written as

$$\begin{bmatrix} \psi^1 - 1 & -\Delta t f \\ \Delta t f & \psi^1 - 1 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

As shown preceding (5-31), this homogeneous set of algebraic equations has a solution only if the determinant of the coefficients is equal to zero, thus

$$(\psi^1 - 1)^2 + (\Delta t)^2 f^2 = \psi^2 - 2\psi^1 + 1 + (\Delta t)^2 f^2 = 0.$$