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An Example of Non-Linear Computational Instability

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Abstract

A particular example is constructed to demonstrate that the finite-difference solution of the non-linear barotropic vorticity equation may have instabilities of a different nature than those caused by either an incorrect choice of the time increment or incorrect lateral boundary conditions. This instability arises because the grid system cannot resolve wave lengths shorter than about 2 grid intervals; when such wave lengths are formed by the non-linear interaction of longer waves, the grid system interprets them incorrectly as long waves. The seemingly successful use of a smoothing process to eliminate this difficulty is described.

1. Introduction

Suppose we are applying the barotropic non-divergent vorticity equation to the two-dimensional flow of an ideal fluid contained in a channel between parallel walls located at $y = 0$ and $y = W$, using the finite-difference methods now employed in numerical weather prediction. To make matters simple, let us restrict the initial flow patterns to those which have both the stream-function ψ and vorticity ζ identically zero on both lateral boundaries ($y = 0$ and $y = W$), and to patterns which are periodic in x, so that $\psi(x, y, 0) \equiv \psi(x \pm L, y, 0)$. It is then clear that these boundary conditions will be valid for all time, and that the flow will maintain its periodic character in x.

We introduce as usual a finite-difference grid,

$$\begin{aligned} x &= j\Delta, & j &= 0, 1, 2, \ldots, J-1. \ (J \text{ even}) \\ y &= k\Delta, & k &= 0, 1, 2, \ldots, K \\ t &= \tau\Delta t, & \tau &= 0, 1, 2 \ldots \end{aligned}$$

where Δ is the space increment and Δt the time increment, and we suppose that W and L are such that $L = J\Delta$ and $W = K\Delta$.

The vorticity equation is

$$\frac{\partial \zeta}{\partial t} = J\left(\frac{\zeta, \psi}{x, y}\right); \quad \zeta = \nabla^2 \psi \qquad (1)$$

(We are not here concerned with the variation of the Coriolis parameter.)

The finite-difference analogue of this which would normally be used, is

$$\nabla^2 (\psi_{\tau+1} - \psi_{\tau-1})_{jk} = \frac{\Delta t}{2\Delta^2} \cdot$$
$$\cdot [\delta_j(\nabla^2\psi) \cdot \delta_k(\psi) - \delta_k(\nabla^2\psi) \cdot \delta_j(\psi)]_{jk\tau} \qquad (2)$$

Here δ_j and δ_k are the usual simple centered difference operators in the x and y directions:

$$\delta_j(\psi) = \psi_{j+1k} - \psi_{j-1k},$$
$$\delta_k(\psi) = \psi_{jk+1} - \psi_{jk-1}.$$

$\nabla^2\psi$ represents the finite-difference approximation for the Laplacian:

$$\Delta^2 \nabla^2\psi \approx \nabla^2\psi =$$
$$= \psi_{j+1k} + \psi_{j-1k} + \psi_{jk+1} + \psi_{jk-1} - 4\psi_{jk} \qquad (3)$$

(2) would be applied at the interior points $j = 0, \ldots, J-1$, and $k = 1, 2, \ldots, K-1$. At the boundary points where $k = 0$ or K, ψ and $\nabla^2\psi$ are both taken to be identically zero for all time. At the points for which $j = 0$ and $j = J-1$, the cyclic condition that $\psi(j, k) \equiv \psi(j \pm J, k)$ would be used.

The streamfunction field defined in this manner at the grid points j, k can then be represented by the finite sum:

$$\psi_{jk\tau} = \sum_{l=0}^{J/2} \sum_{m=1}^{K-1} \left[a_{lm\tau} \cos\frac{2\pi jl}{J} + b_{lm\tau} \sin\frac{2\pi jl}{J} \right] \cdot \sin\frac{\pi mk}{K}, \qquad (4)$$

We may investigate the magnitude of the unavoidable instability as follows. From the form of ψ_2, we can think of A and B as equal to $\frac{1}{2}v\Delta$, where v is the velocity due to the difference in ψ_2 at the points $(j+1, k)$ and (j, k). Introducing this definition we find

$$\cosh \Theta = 1 + \frac{3}{200}\left(\frac{v\Delta t}{\Delta}\right)^2.$$

The ordinary linearized analysis of computational stability would have led us to a choice of $(v\Delta t/\Delta)$ less than 1. For small $(v\Delta t/\Delta)$, $\Theta \approx 0.17 \cdot \cdot (v\Delta t\Delta) \sim 0.1$, say, and only 20 time steps will produce an e-fold amplification.

The above analysis is of course only a very particular example. A more thorough analysis of the general case is perhaps too difficult to make, but it seems unlikely that the mere presence of more than 3 degrees of freedom would remove this instability.

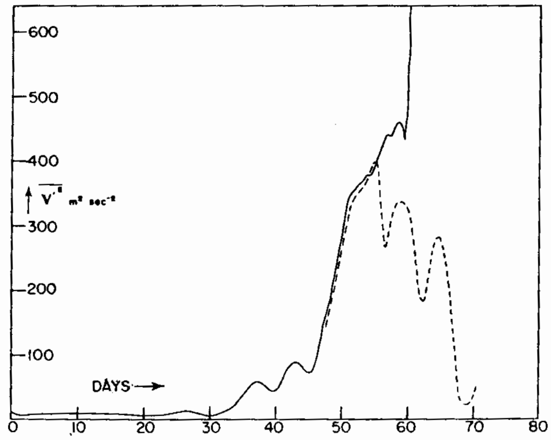

Fig. 1. Disturbance kinetic energy as a function of time. The solid curve was obtained without smoothing, the computations breaking down at about 56 days. The dashed curve was obtained by periodically introducing a filtering procedure.

3. Elimination of the instability by smoothing

Several years ago, the writer applied the techniques of numerical prediction to the study of the general circulation of the atmosphere (PHILLIPS, 1956). This was done by making a forecast for an extended period with a 2-level geostrophic model. The equations included a crude representation of heating and friction, and were applied to a simplified geometrical model of the atmosphere—the so-called "β-plane". After a period of several weeks, the appearance of large truncation errors caused an almost explosive increase in the total energy of the system.

In an attempt to explore this type of computation error, a similar set of equations has recently been solved again, using a smaller horizontal grid interval —$166\,^2/_3$ km compared to the grid intervals of $\Delta x = 375$ km and $\Delta y = 625$ km in the earlier experiment. Although enough changes were also made in the differential equations (e.g. the representation of friction and of the heating function) to prevent a complete comparison between the two computations with respect to truncation error, the same catastrophic errors appeared again, and at a time when the disturbance kinetic energy was about the same magnitude as it was when the breakdown occurred in the first computation. Thus, the reduction in the grid distance, Δ, which should have reduced the truncation error, did not appreciably postpone the breakdown.

The graph of $\overline{\mathbf{v}'^2}$ —proportional to the kinetic energy per unit mass of the disturbance—for this second computation is shown by the full line in fig. 1. The truncation errors became significant around 56 days, just prior to the explosive increase in $\overline{\mathbf{v}'^2}$.[1]

The dashed line in the figure is the curve obtained by redoing the computations (beginning at 48 days) and *periodically eliminating all components with wave lengths smaller than* 4Δ. This was accomplished by performing a Fourier analysis on the grid point data every Nth time step, and then reconstituting the smoothed field, retaining only wave numbers $l = 0, 1, \ldots, J/4$, and $m = 1, 2, \ldots, K/2$. (N was variously chosen so as to give either a 2-hr or 6-hr interval between smoothing operations, little difference being found in the results for the two intervals.)

This smoothed forecast satisfied the energy budgets very well. Let δ be the difference between (a) the observed change in total energy over a one day interval and (b) the theoretical change in total energy computed from the gains due to non-adiabatic heating and the losses due to friction. The root mean square value of δ during the smoothed forecast (48—70 days) was only 0.23 joules sec^{-1} m^{-2}, and the mean value of δ was close to zero. (This was also typical of the value in the unsmoothed forecast *before*

[1] For example the *difference* between the observed change in total energy over one day and the change computed from the energy transformation integrals first exceeded 1 joule sec^{-1} m^{-2} at 55 days.

the sudden breakdown in that forecast at around 56 days.) This suggests that these geostrophic equations do not readily transmit energy to horizontal wave lengths shorter than 700 km— a result already familiar from the analysis by Fjørtoft (Fjørtoft, 1953) — since otherwise the smoothing process would have taken a noticeable amount of energy out of the system. However, the discussion above of the non-linear instability mechanism, and the success of the smoothing procedure, together indicate that even this small rate of energy transfer may be sufficient to activate non-linear computational instabilities in wave lengths shorter than 4 grid intervals if these components are not artificially removed.

In conclusion it may be appropriate to point out that misrepresentation errors similar to (5) will be encountered in solving the non-linear "balance equation" by finite differences (Bolin, 1955; Charney, 1955). This has already been noted by Shuman, who has developed some useful approximations to the straightforward but time consuming Fourier smoothing (Shuman, 1957).

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where the coefficients $a_{lm\tau}$ and $b_{lm\tau}$ are functions of τ. In this formulation, we take $b_{0m\tau} = b_{J/2m\tau} \equiv 0$, so that there are $J(K-1)$ degrees of freedom in the grid point values of ψ_{jk} and also in the coefficients a_{lm} and b_{lm}. We see from this representation that the smallest wave length in x recognized by the grid system is for $l = J/2$ and corresponds to a wave length of 2Δ. In y, the smallest wave length is for $m = K-1$, and corresponds to a wave length of $2\Delta(K-1)/K \sim 2\Delta$.

Equation (2) is non-linear. If we consider the interaction of 2 components ψ_1 and ψ_2, which are characterized by the wave numbers (l_1, m_1) and (l_2, m_2), it can be seen from (2) that they will contribute to the time rate of change of the 4 components with wave numbers (l_1+l_2, m_1+m_2), (l_1+l_2, m_1-m_2), (l_1-l_2, m_1+m_2), and (l_1-l_2, m_1-m_2). This non-linear interaction determines the transfer of kinetic energy between different parts of the spectrum in this type of flow, and, in the meteorological problem, becomes very important when forecasts are to be made for any extended period of time.

We now recall that any distribution of ψ on the grid network jk can be resolved into the Fourier sum (4), containing only wave numbers $l = 0, 1, \ldots, J/2$ and $m = 1, 2 \ldots, K-1$. It is then clear that the interaction of ψ_1 and ψ_2 with each other will not be interpreted correctly when $l_1 + l_2 > J/2$ and/or when $m_1+m_2 > K-1$. For example, if $l_1+l_2 = J-r$, with $r < J/2$, we would find the following type of misrepresentation to occur:

$$\left.\begin{aligned} \cos\frac{2\pi j}{J}(l_1+l_2) &= \cos\frac{2\pi j}{J}(J-r) = \cos\frac{2\pi j}{J}r, \\ \sin\frac{2\pi j}{J}(l_1+l_2) &= \sin\frac{2\pi j}{J}(J-r) = -\sin\frac{2\pi j}{J}r. \end{aligned}\right\} \tag{5}$$

Thus, instead of affecting wave number $J-r$, the components ψ_1 and ψ_2 will affect wave number r. A similar misrepresentation will occur in the m wave numbers whenever $m_1+m_2 > K-1$.

2. An example of instability from this source

The potential seriousness of this misrepresentation can be seen by constructing an artificial example.

We take only 2 components:

$$\left.\begin{aligned} \psi_1 &= \left[C_\tau \cos\frac{\pi j}{2} + S_\tau \sin\frac{\pi j}{2}\right]\sin\frac{2\pi k}{3}, \\ \psi_2 &= U_\tau \cos \pi j \sin\frac{2\pi k}{3}, \end{aligned}\right\} \tag{6}$$

Thus $l_1 = J/4$, $m_1 = 2K/3$, and $l_2 = J/2$, $m_2 = 2K/3 = m_1$. The misrepresentation which occurs is of the form

$$\left.\begin{aligned} l_1 + l_2 &= \frac{3J}{4} = J - \frac{J}{4} = J - l_1, \text{ and} \\ m_1 + m_2 &= \frac{4K}{3} = K - \frac{2K}{3} = K - m_1. \end{aligned}\right\} \tag{7}$$

Since $l_1 - l_2$ in this case is equal to l_1, and $m_1 - m_2$ is equal to zero, no new harmonics are generated by the finite-difference interaction of ψ_1 and ψ_2. The *exact* finite-difference solution of this particular example is then described by the three ordinary non-linear difference equations:

$$\left.\begin{aligned} C_{\tau+1} - C_{\tau-1} &= \sigma U_\tau S_\tau, \\ S_{\tau+1} - S_{\tau-1} &= \sigma U_\tau C_\tau, \\ U_{\tau+1} - U_{\tau-1} &= 0, \\ (\sigma = \sqrt{3}&\Delta t/5\Delta^2). \end{aligned}\right\} \tag{8}$$

These are the result of inserting (6) into (2) and (3). Although non-linear, they are simple enough to be solved. We first find that U_τ has the constant value A for even τ and the constant value B for odd τ. C_τ (or S_τ) then satisfies the difference equation:

$$C_{\tau+2} - 2\cosh\Theta\, C_\tau + C_{\tau+2} = 0,$$

where $\cosh\Theta = 1 + \frac{1}{2}\sigma^2 AB$ is a constant. This difference equation has four solutions:

$$e^{\frac{\Theta\tau}{2}},\ (-1)^\tau e^{\frac{\Theta\tau}{2}},\ e^{-\frac{\Theta\tau}{2}},\ (-1)^\tau e^{-\frac{\Theta\tau}{2}}$$

If A and B have the same sign, Θ is a real number, and two of the solutions will amplify exponentially. *This "instability" cannot be eliminated by reducing Δt.*

If A and B are of opposite sign, but small enough in magnitude so that $1 + \frac{1}{2}\sigma^2 AB > -1$, Θ is pure imaginary and we have four neutral solutions. However, if A and B are of opposite sign but large enough in magnitude so that $1 + \frac{1}{2}\sigma^2 AB < -1$, the solutions are again of the form $\exp \pm \Phi\tau/2$ where $\cosh\Phi = |1 + \frac{1}{2}\sigma^2 AB|$. These again will amplify with time, since Φ will be real. Thus, when A and B are of opposite sign, the instability can be eliminated by reducing Δt.