Chapter 4. Nonlinear Hyperbolic Problems

4.4 System of Hyperbolic Equations – Shallow Water Equation model

Ref: Chapter IV of Mesinger and Arakawa (1976) GARP Report

4.4.1. Introduction

It is assumed that you are familiar with the shallow water equations and associated theories. If not, consult Holton or Haltiner and Williams book.

The following is a set of linear 1D shallow water equations:

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + \frac{\partial \phi'}{\partial x} = 0 \qquad \text{(21a)}$$

$$\frac{\partial \phi'}{\partial t} + \bar{u}\frac{\partial \phi'}{\partial x} + \Phi\frac{\partial u'}{\partial x} = 0 \qquad \text{(21b)}$$

$\bar{u}$ = constant base state flow

$\Phi = gH$ = $g \times$ mean depth of the water = constant

$u \rightarrow u'$ = perturbation velocity

$\phi = gh'$ = perturbation geopotential height

<u>Issues to consider with respect to numerical solution</u>

1) More than 1 variable
2) Equations coupled
3) Can support multiple physical modes
4) There are more possibilities of grid layout (see figure below)

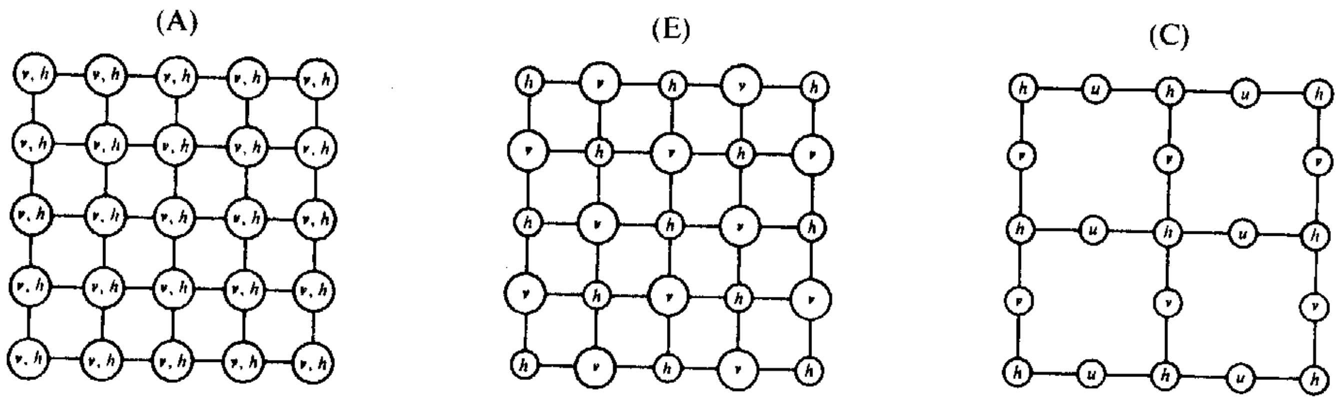

Figure 2.1 Three types of lattice considered for the finite difference solution of 2.1.

4.4.2. The differential solution

Performing standard analysis by assuming

$$\psi = \Psi \exp[i(kx - \omega t)] \tag{22}$$

gives $$\omega = k(\bar{u} \pm \sqrt{\Phi}) \tag{23}$$

which is called the dispersion relation.

From (23) →

$$c = \frac{\omega}{k} = \bar{u} \pm \sqrt{\Phi}\ .$$

In the phase speed, there are slow mode represented by $\bar{u}$ (advection) and fast mode given by $\sqrt{\Phi}$ (surface gravity waves). Since c is constant, the waves are <u>non-dispersive</u>.

Group velocity

$$c_g = \frac{\partial \omega}{\partial k} = \bar{u} \pm \sqrt{\Phi}$$

represents the speed of wave energy propagation.

What about the characteristics (we have seen this before – see example problem given at the end of Chapter 1). Make use of the auxiliary equations, we have the following equations in matrix form:

$$\begin{bmatrix} 1 & \bar{u} & 0 & 1 \\ 0 & \Phi & 1 & \bar{u} \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{bmatrix} \begin{pmatrix} u_t \\ u_x \\ \phi_t \\ \phi_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ du \\ d\phi \end{pmatrix} \qquad (24)$$

Setting the determinant of the coefficient matrix to zero gives

$$\left(\frac{dx}{dt}\right)^2 - 2\overline{u}\frac{dx}{dt} + (\overline{u}^2 - \Phi) = 0 \rightarrow$$

$$\frac{dx}{dt} = \overline{u} \pm \sqrt{\Phi}$$

which are the characteristics equations.

The compatibility equations can be found to be

$$u \pm \frac{\phi}{\sqrt{\Phi}} = \text{constant along } \frac{dx}{dt} = \overline{u} \pm \sqrt{\Phi}\,. \qquad (25)$$

Eq. (25) can be rewritten as

$$\left[\frac{\partial}{\partial t} + \left(\overline{u} + \sqrt{\Phi}\right)\frac{\partial}{\partial x}\right]\left(\overline{u} + \frac{\phi}{\sqrt{\Phi}}\right) = 0 \qquad (26a)$$

$$\left[\frac{\partial}{\partial t} + \left(\overline{u} - \sqrt{\Phi}\right)\frac{\partial}{\partial x}\right]\left(\overline{u} - \frac{\phi}{\sqrt{\Phi}}\right) = 0 \qquad (26b)$$

which are two <u>decoupled equations</u> describing wave disturbances 'advected' by the respective propagation speeds. $\overline{u} \pm \phi / \sqrt{\Phi}$ are known as the <u>Riemann invariants</u>, as said before.

Equations (26) can also be obtained using matrix method (see Chapter 1).

4.4.3. Discretization for the Shallow Water Equations

4.4.3.1. Forward-backward scheme

We know that FTCS is unstable for pure advection equations, and this is also true to the shallow water equations.

But, we can obtain a stable scheme if we use backward scheme for the second equation. Let's look at the simper case of $\bar{u}$=0, i.e., there is not mean flow:

$$\begin{aligned} &\delta_{+t}u + \delta_{2x}\phi^{n} = 0 \\ &\delta_{+t}\phi + \Phi\delta_{2x}u^{n+1} = 0 \end{aligned} \qquad (27)$$

Since forward scheme is used for the first equation and backward scheme used for the second, the overall scheme is called forward-backward scheme. We can show that it is conditionally stable.

Stability Analysis

Assume that

$$\begin{aligned} &u_j^n = A\lambda^n \exp(ikx_j) \\ &\phi_j^n = B\lambda^n \exp(ikx_j) \end{aligned} \qquad (28)$$

Note here A and B could be complex so as to allow possible phase difference between u and ϕ.

Plug (28) into (27) →

$$(\lambda^{n+1}-\lambda^{n})A+B\lambda^{n}\frac{\Delta t}{2\Delta x}(e^{ik\Delta x}-e^{-ik\Delta x})=0$$
$$(\lambda^{n+1}-\lambda^{n})B+\Phi A\lambda^{n+1}\frac{\Delta t}{2\Delta x}(e^{ik\Delta x}-e^{-ik\Delta x})=0 \tag{29}$$

or

$$(\lambda-1)A+iB\frac{\Delta t}{\Delta x}\sin(k\Delta x)=0$$
$$(\lambda-1)B+i\Phi A\frac{\Delta t}{\Delta x}\lambda\sin(k\Delta x)=0 \tag{30}$$

or

$$\begin{pmatrix} \lambda-1 & i\frac{\Delta t}{\Delta x}\sin(k\Delta x) \\ i\Phi\frac{\Delta t}{\Delta x}\lambda\sin(k\Delta x) & \lambda-1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{30'}$$

(30') is a simultaneous linear system of equations for *A* and *B*. It has non-trivial solutions if and only if the determinant of the coefficient matrix equals to zero. →

$$\lambda^2-\lambda[2-\Phi a^2]+1=0 \quad [\text{where } a=\Delta t/\Delta x\sin(k\Delta x)]$$

$$\lambda_{\pm}=\frac{\Phi a^2-2\pm\sqrt{(2-\Phi a^2)^2-4}}{2} \tag{31}$$

If the radical is negative, then $|\lambda_{\pm}|\equiv 1$. I.e., if

$$(2-\Phi a^2)^2 \le 4$$

$$|2-\Phi a^2| \le 2$$

$$-2 \le 2-\Phi a^2 \le 2$$

$2-\Phi a^2 \le 2$ is always satisfied, in addition,

$\Phi a^2 \le 4$ →

$\Delta t \le \left| \dfrac{2\Delta x}{\sqrt{\Phi}\sin(k\Delta x)} \right|$, for it to be valid for all *k*, we require

$$\Delta t \le \frac{2\Delta x}{\sqrt{\Phi}} \quad \text{or} \quad \mu = \frac{\Delta t\sqrt{\Phi}}{\Delta x} \le 2 \tag{32}$$

which is the stability condition! Here $\sqrt{\Phi}$ is the disturbance propagation speed in the absence of base-state advective flow.

When the mean flow is non-zero, and when the advection terms are treated using the upstream-in-space, forward-in-time scheme, the condition should be

$$\Delta t \le \frac{\Delta x}{|\bar{u}| + \sqrt{\Phi}/2}.$$

Note the factor of 2 in the condition – the use of forward-backward scheme actually allows a Courant number of 2 relatively to the gravity wave to be used! This is due to the fact the backward scheme is actually kind of 'implicit' scheme.

4.4.3.2. Centered-in-time (leapfrog) Center-in-Space (CTCS) scheme

$$\delta_{2t}u + \bar{u}\delta_{2x}u + \delta_{2x}\phi = 0$$
$$\delta_{2t}\phi + \bar{u}\delta_{2x}\phi + \Phi\delta_{2x}u = 0$$

(here we assume a non-staggered grid)

Similar stability analysis leads to:

$$\Delta t \le \frac{\Delta x}{|\bar{u}| + \sqrt{\Phi}} \qquad (33)$$

which is twice as restrictive as that for forward-backward scheme. Also it contains a computational mode.

Grid Splitting

The above CTCS scheme used non-staggered grid.

When using non-staggered grid for the above equations, we can run into the grid-splitting problem. We discussed this issue in the past.

One way of avoiding grid splitting is to use staggered grid – in which different variables are located at different points of a grid mesh.

Let's stagger u and ϕ (h in the figure) in the following way:

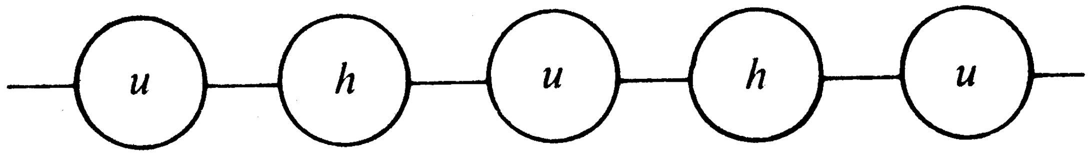

Figure 1.2 A grid with two dependent variables that are carried at alternate grid points.

Our FD equation using CTCS scheme is then

$$\begin{aligned} &\delta_{2t}u + \bar{u}\delta_{2x}u + \delta_x\phi = 0 \qquad &\text{at u point} \\ &\delta_{2t}\phi + \bar{u}\delta_{2x}\phi + \Phi\delta_x u = 0 \qquad &\text{at } \phi \text{ point} \end{aligned} \tag{34}$$

Note the key difference in the third term of each equation from the previous non-staggered CTCS scheme. Also the equations are solved at different grid point.

Stability analysis will show that the stability condition is

$$\Delta t \le \frac{\Delta x}{|\bar{u}| + 2\sqrt{\Phi}}$$

which, for zero mean flow case, is twice as restrictive as the non-staggered version (because of the factor of 2 in front of phase speed $\sqrt{\Phi}$).

However, since the pressure gradient force and velocity divergence terms are differenced over one Δx interval instead of over $2\Delta x$, and these are terms responsible for the gravity wave propagation, the solution should be more accurate, since the effective grid spacing is half as large.

4.4.3.3. Treatment of insignificant fast modes

(Reading: Durran Chapter 7 – Physically insignificant fast waves)

We obtained earlier the phase speed of shallow water waves:

$$c = \bar{u} \pm \sqrt{gH}$$

it contains two modes. The slower advective mode and the faster gravity wave (GW) mode:

$$\bar{u} \sim 10 m/s$$

$\sqrt{gH} \sim \sqrt{10 \times 10000} \sim 200 m/s$ for external gravity waves

$|\bar{u}| \ll \sqrt{gH}$ for many problems.

Gravity waves are not important in global coarse-resolution (effective grid spacing > 100 km) NWP models in which the resolutions are usually too coarse to resolve them adequately anyway.

GWs are often important for mesoscale (~100 km in scale) atmospheric flows. For mesoscale NWP models, often, compressible equations are used which support fast sound waves – so sound wave play a similar role as the gravity waves in large scale model in limiting the time step size (when using explicit schemes).

When the fast mode is not important, we don't want it to be the one that limits the time step size.

There are in general two ways to deal with this problem – one is to treat the terms responsible for the fast modes implicitly, and the other uses different time step sizes for fast and slow modes and the method is called mode splitting method. ARPS, MM5 and WRF models use the latter to deal with fast sound waves (hence the large and small time steps, dtbig and dtsml you find in arps.input).

4.4.4.4. Semi-implicit method

Since the PGF term in u equation and the velocity divergence term in ϕ equation are responsible for gravity waves, we can treat them implicitly, so that hopefully the gravity wave mode no longer limit the time step size.

Again we look at the non-staggered case:

$$\begin{aligned} &\delta_{2t}u + \bar{u}\delta_{2x}u + \overline{\delta_{2x}\phi}^{2t} = 0 \\ &\delta_{2t}\phi + \bar{u}\delta_{2x}\phi + \Phi\overline{\delta_{2x}u}^{2t} = 0 \end{aligned} \qquad (35)$$

The time averages makes the scheme implicit. Since only some of the terms are treated implicitly, the scheme is called semi-implicit.

Stability of the system – only the advective velocity $\bar{u}$ appears in the stability condition therefore much larger time step can be used (see Durran 7.2.3; Mesinger and Arakawa Chapter 4 section 6).

Analysis shows that the fast mode in the numerical solution is actually slowed down – i.e., there is a lagging phase error with this mode – it is okay if this mode is consider unimportant, like the sound waves in the atmosphere or the gravity waves in large-scale models.

Solution procedure for (35)

1) Computer ϕ^{n+1} for all j by eliminating u^{n+1} from the 2nd equation using the first:

$$\phi_j^{n+1} - \frac{\Phi \Delta t^2}{4\Delta x^2}\left[\phi_{j-2}^{n+1} - 2\phi_j^{n+1} + \phi_{j+2}^{n+1}\right] = f \ ,$$

with the right hand side being known.

2) Two effectively decoupled tridiagonal systems of equations (for 1D problems) have to be solved, one for even *j* and one for odd *j* (can lead to grid splitting).

3). Once ϕ^{n+1} is known, we can plug it into *u* equation to obtain u^{n+1}.

4) If a staggered grid is used, then only one tridiagonal system of equations has to be solved. The total amount of calculation is about the same as the non-staggered case because twice any many grid points are now involved.

5) For 2D or 3D problems, the semi-implicit scheme results in a Helmholtz equation that can't be as easily solved as the 1D tridiagonal equation.

   Tapp and White is one of the first to use semi-implicit method in a compressible mesoscale model of the UK Met Office (Tapp and White 1976 QJRMS).

Derivation of *f* in the above equation:

$$\delta_{2t}u + \bar{u}\delta_{2x}u + \overline{\delta_{2x}\phi}^{2t} = 0$$

$$u^{n+1} = u^{n-1} - 2\Delta t\bar{u}\delta_{2x}u - 2\Delta t\overline{\delta_{2x}\phi}^{2t}$$

$$\delta_{2x}u^{n+1} = \delta_{2x}u^{n-1} - 2\Delta t\bar{u}\delta_{2x}(\delta_{2x}u) - 2\Delta t\delta_{2x}(\overline{\delta_{2x}\phi}^{2t})$$

$$\delta_{2t}\phi + \bar{u}\delta_{2x}\phi + \Phi\overline{\delta_{2x}u}^{2t} = 0$$

$$\delta_{2t}\phi + \bar{u}\delta_{2x}\phi + \frac{1}{2}\Phi\left[(\delta_{2x}u)^{n+1} + (\delta_{2x}u)^{n-1}\right] = 0$$

$$\begin{aligned}\phi^{n+1} &= \phi^{n-1} - 2\Delta t\bar{u}\delta_{2x}\phi - \Delta t\Phi\left[(\delta_{2x}u)^{n+1} + (\delta_{2x}u)^{n-1}\right] \\ &= \phi^{n-1} - 2\Delta t\bar{u}\delta_{2x}\phi - \Delta t\Phi\left[\delta_{2x}u^{n-1} - 2\Delta t\bar{u}\delta_{2x}(\delta_{2x}u) - 2\Delta t\delta_{2x}(\overline{\delta_{2x}\phi}^{2t}) + (\delta_{2x}u)^{n-1}\right]\end{aligned}$$

$$\begin{aligned}&\phi^{n+1} - (\Delta t)^2\Phi\delta_{2x}(\delta_{2x}\phi)^{n+1} \\ &\quad = \phi^{n-1} - 2\Delta t\bar{u}\delta_{2x}\phi - \Delta t\Phi\left[2\delta_{2x}u^{n-1} - 2\Delta t\bar{u}\delta_{2x}(\delta_{2x}u) - \Delta t\delta_{2x}(\delta_{2x}\phi)^{n-1}\right]\end{aligned}$$

$$\begin{aligned}&\phi^{n+1} - (\Delta t)^2\Phi\delta_{2x}(\delta_{2x}\phi)^{n+1} \\ &\quad = \phi^{n-1} - 2\Delta t\left[\bar{u}\delta_{2x}\phi + \Phi\delta_{2x}u^{n-1}\right] + (\Delta t)^2\left[2\bar{u}\delta_{2x}(\delta_{2x}u) + \delta_{2x}(\delta_{2x}\phi)^{n-1}\right]\end{aligned}$$

$f = \phi^{n-1} - 2\Delta t\left[\bar{u}\delta_{2x}\phi + \Phi\delta_{2x}u^{n-1}\right] + (\Delta t)^2\left[2\bar{u}\delta_{2x}(\delta_{2x}u) + \delta_{2x}(\delta_{2x}\phi)^{n-1}\right]$ in the above.

Note that $\delta_{2x}(\delta_{2x}\phi) \equiv \dfrac{\phi_{j+2} - 2\phi_j + \phi_{j-2}}{(2\Delta x)^2}$ and terms without superscript are evaluated at time level *n*.

4.4.4.5. Mode-splitting Method

Another way of dealing with fast modes in a modeling system is to use the 'mode-splitting' method. Klemp and Wilhemson (1978) were the first to apply such a method to a full atmospheric model, a 3D compressible cloud model. In the following, we look at (2D) equations for a compressible atmospheric that are more like those used in the Advanced Regional Prediction System (Xue et al. 2000 MAP):

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}+D_u$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z}-\frac{gp'}{\gamma\bar{p}}+g\frac{\theta'}{\bar{\theta}}+D_w$$

$$\frac{\partial p'}{\partial t}+u\frac{\partial p'}{\partial x}+w\frac{\partial p'}{\partial z}-\bar{\rho}gw+\bar{\rho}c_s^2\left[\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right]=0$$

$$\frac{\partial \theta'}{\partial t}+u\frac{\partial \theta'}{\partial x}+w\frac{\partial \theta'}{\partial z}+w\frac{\partial \bar{\theta}}{\partial z}=\dot{S}$$

They are, of course, the horizontal and vertical momentum equations, the pressure equation and potential temperature equations. The pressure equation is derived from mass continuity equation for density employing the equation of state. Direct adiabatic equation on pressure is neglected. The base state variables (with overbar) are assumed to be horizontally homogeneous and hydrostatic.

The above equations are be re-written as the follow, where *UADV*, *WADV*, *PADV* and *TADV* represent the advection terms for u, w, p and potential temperature, respectively.

$$\frac{\partial u}{\partial t}+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}=-UADV+D_u=F_u+D_u$$

$$\frac{\partial w}{\partial t}+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z}+\frac{gp'}{\gamma\bar{p}}=-WADV+g\frac{\theta'}{\bar{\theta}}+D_w=F_w+D_w$$
$$\frac{\partial p'}{\partial t}+\bar{\rho}gw-\bar{\rho}c_s^2\left[\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right]=-PADV=F_p$$
$$\frac{\partial \theta'}{\partial t}=-TADV+\dot{S}+D_\theta=F_\theta+D_\theta$$

In the above equations, all terms related to the fast acoustic modes are placed on the left hand side of the equations, while not connected with the acoustic modes, i.e., those corresponding to the 'slow' modes in the system are placed on the right hand side and denoted *F*.

Leapfrog scheme is used for the large time step (Δt) terms. Forward-backward scheme is used for the horizontal acoustic waves (the horizontal divergence term in *p'* equation) in the small time step ($\Delta\tau$) integration. Vertical acoustic waves are treated implicitly, by calculating the vertical pressure gradient term and vertical divergence term in w and p equations, respectively, as their average between the current (τ) and future time ($\tau+\Delta\tau$) levels.

Here $2\Delta t=m\Delta\tau$, *m* is an integer larger or equal to 1. In some situations (e.g., when large *w* prevents the use of large Δt while $\Delta\tau$ is not subject to the small *dz* limitation due to the use of implicit scheme in the vertical), we might be allowed to use $\Delta\tau=2\Delta t$, that is, the 'small' time step is larger than the 'larger' time step.

$$\frac{u^{\tau+\Delta\tau}-u^{\tau}}{\Delta\tau}=-\frac{1}{\bar{\rho}}\delta_x p'^{\tau}+F_u^t+D_u^{t-\Delta t},$$

$$\frac{w^{\tau+\Delta\tau}-w^{\tau}}{\Delta\tau}=-\frac{1}{\bar{\rho}}\left[0.5\delta_z p'^{\tau}+0.5\delta_z p'^{\tau+\Delta\tau}\right]-\frac{gp'^{\tau}}{\gamma\bar{p}}+F_w^t+D_w^{t-\Delta t},$$

$$\frac{p'^{\tau+\Delta\tau} - p'^{\tau}}{\Delta\tau} = -\bar{\rho} g w^{\tau} + \bar{\rho} c_s^2 \left[\delta_x u^{\tau+\Delta\tau} + 0.5\delta_z w^{\tau} + 0.5\delta_z w^{\tau+\Delta\tau} \right] + F_p^t,$$

$$\frac{\theta'^{\tau+\Delta\tau} - \theta'^{\tau}}{\Delta\tau} = F_\theta^t + D_\theta^{t-\Delta t},$$

where the small time steps start at $t-\Delta t$ and end at $t+\Delta t$. The diffusion terms D are evaluated at the past time level to make the large time step integration 'forward' relative to these terms for conditionally stability. Actually, in the ARPS, the vertical diffusion terms are treated implicitly also to avoid strong limitation on Δt when vertical diffusion is very strong in convectively unstable boundary layer (where and when the vertical mixing coefficient is large from the PBL scheme).

Skamarock and Klemp (1992) discuss that stability issues associated with the mode-splitting methods as applied to compressible system of equations. See also Klemp and Wilhemson (1978).

References:

Klemp, J. B., and R. B. Wilhelmson, 1978: The simulation of three-dimensional convective storm dynamics. *J. Atmos. Sci.*, **35**, 1070-1096.

Skamarock, W. C., and J. B. Klemp, 1992: The stability of time-split numerical methods for the hydrostatic and nonhydrostatic elastic equations. *Mon. Wea. Rev.*, **120**, 2109-2127.

Xue, M., K. K. Droegemeier, and V. Wong, 2000: The Advanced Regional Prediction System (ARPS) - A multiscale nonhydrostatic atmospheric simulation and prediction tool. Part I: Model dynamics and verification. *Meteor. Atmos. Phy.*, **75,** 161-193.

4.4.4. The Arakawa Grids

(p.47 in Mesinger and Arakawa 1976)

Arakawa (Arakawa and Lamb 1977) introduced a variety of staggered grids when trying to find the most accurate method for handling geostrophic adjustment process, which we know relies on inertia gravity waves. Inertia gravity waves are dispersive, they disperse ageostrophic energy.

To describe inertia GW, we need to include rotational effect into the shallow water equations:

$$\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} - fv = 0$$
$$\frac{\partial v}{\partial t} + g\frac{\partial h}{\partial y} + fu = 0$$
$$\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

For 1-D version of this problem, i.e. for $\frac{\partial}{\partial y} = 0$ case, the dispersion equation for the exact solution is

$$\omega = (f^2 + k^2 gH)^{1/2}.$$

Arakawa defined 5 different grids, all of which has the same number of dependent variables per unit area – so that the computational time is about the same.

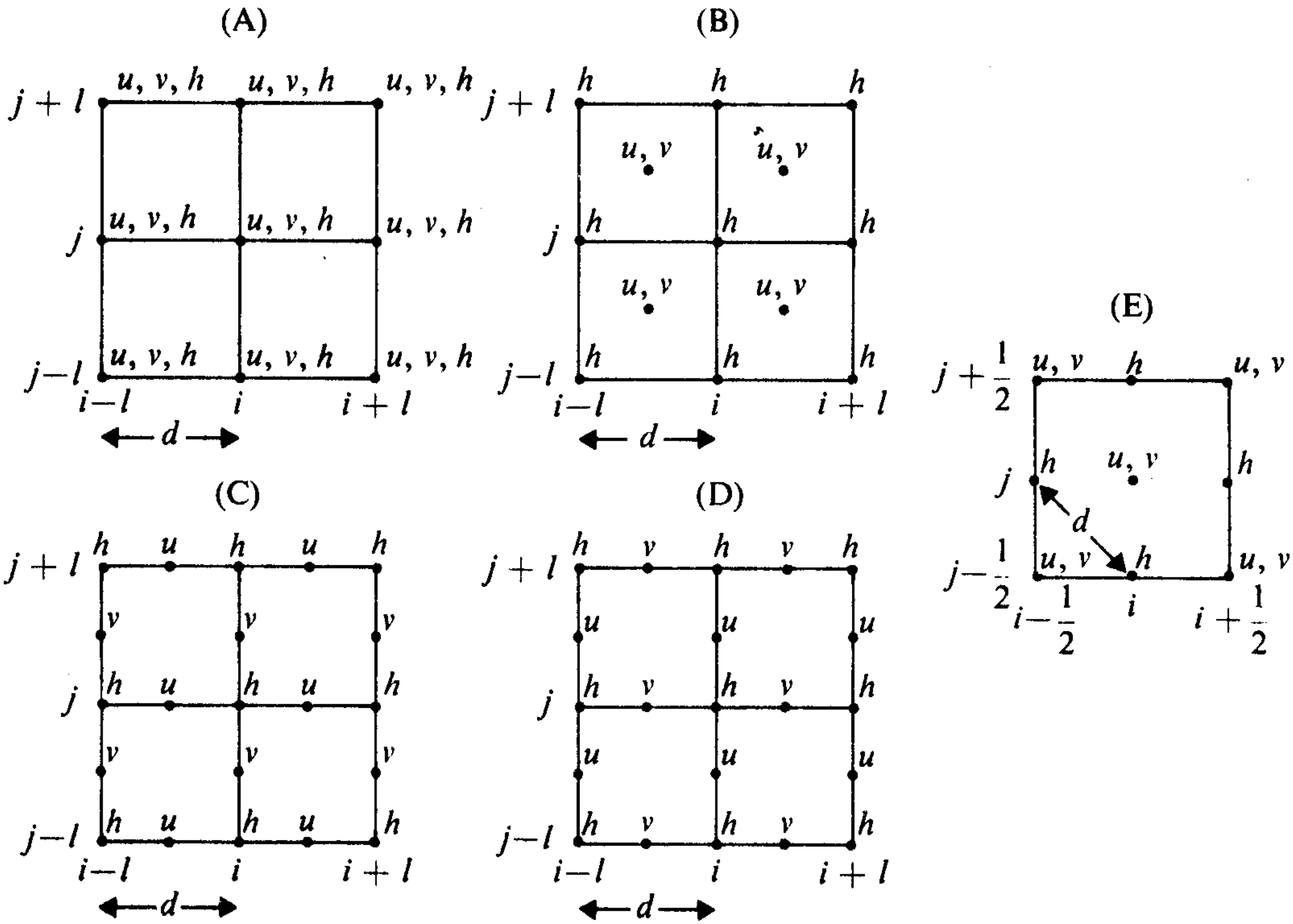

Figure 3.1 Five types of lattice considered for the finite difference solution of (3.1).

For each of the above grid, the finite difference equation can be written as

$$\frac{\partial u}{\partial t} = -g\,\overline{\delta_x h}^{x} + fv, \quad \frac{\partial v}{\partial t} = -g\,\overline{\delta_y h}^{y} - fu,$$
$$\frac{\partial h}{\partial t} = -H\left(\overline{\delta_x u}^{x} + \overline{\delta_y v}^{y}\right); \tag{3.2$_A$}$$

$$\frac{\partial u}{\partial t} = -g\,\overline{\delta_x h}^{y} + fv, \quad \frac{\partial v}{\partial t} = -g\,\overline{\delta_y h}^{x} - fu,$$
$$\frac{\partial h}{\partial t} = -H\left(\overline{\delta_x u}^{y} + \overline{\delta_y v}^{x}\right); \tag{3.2$_B$}$$

$$\frac{\partial u}{\partial t} = -g\,\delta_x h + f\bar{v}^{xy}, \quad \frac{\partial v}{\partial t} = -g\,\delta_y h - f\bar{u}^{xy},$$
$$\frac{\partial h}{\partial t} = -H\left(\delta_x u + \delta_y v\right); \tag{3.2$_C$}$$

$$\frac{\partial u}{\partial t} = -g\,\overline{\delta_x h}^{xy} + f\bar{v}^{xy}, \quad \frac{\partial v}{\partial t} = -g\,\overline{\delta_y h}^{xy} - f\bar{u}^{xy},$$
$$\frac{\partial h}{\partial t} = -H\left(\overline{\delta_x u}^{xy} + \overline{\delta_y v}^{xy}\right); \tag{3.2$_D$}$$

$$\frac{\partial u}{\partial t} = -g\,\delta_x h + fv, \quad \frac{\partial v}{\partial t} = -g\,\delta_y h - fu,$$
$$\frac{\partial h}{\partial t} = -H\left(\delta_x u + \delta_y v\right). \tag{3.2$_E$}$$

We want to find the numerical dispersion relations and compare them with the exact solution. For 1-D problem, the dispersion relations are (note ν here is our ω, $d = \Delta x$, the time derivative terms are not differenced, i.e., remain in their continuous form):

$$\left(\frac{\nu}{f}\right)^2 = 1 + \left(\frac{\lambda}{d}\right)^2 \sin^2 kd, \qquad (3.6)_A$$

$$\left(\frac{\nu}{f}\right)^2 = 1 + 4\left(\frac{\lambda}{d}\right)^2 \sin^2 \frac{kd}{2}, \qquad (3.6)_B$$

$$\left(\frac{\nu}{f}\right)^2 = \cos^2 \frac{kd}{2} + 4\left(\frac{\lambda}{d}\right)^2 \sin^2 \frac{kd}{2}, \qquad (3.6)_C$$

$$\left(\frac{\nu}{f}\right)^2 = \cos^2 \frac{kd}{2} + \left(\frac{\lambda}{d}\right)^2 \sin^2 kd, \qquad (3.6)_D$$

$$\left(\frac{\nu}{f}\right)^2 = 1 + 2\left(\frac{\lambda}{d}\right)^2 \sin^2 \frac{kd}{\sqrt{2}}. \qquad (3.6)_E$$

They are plotted in the following figure:

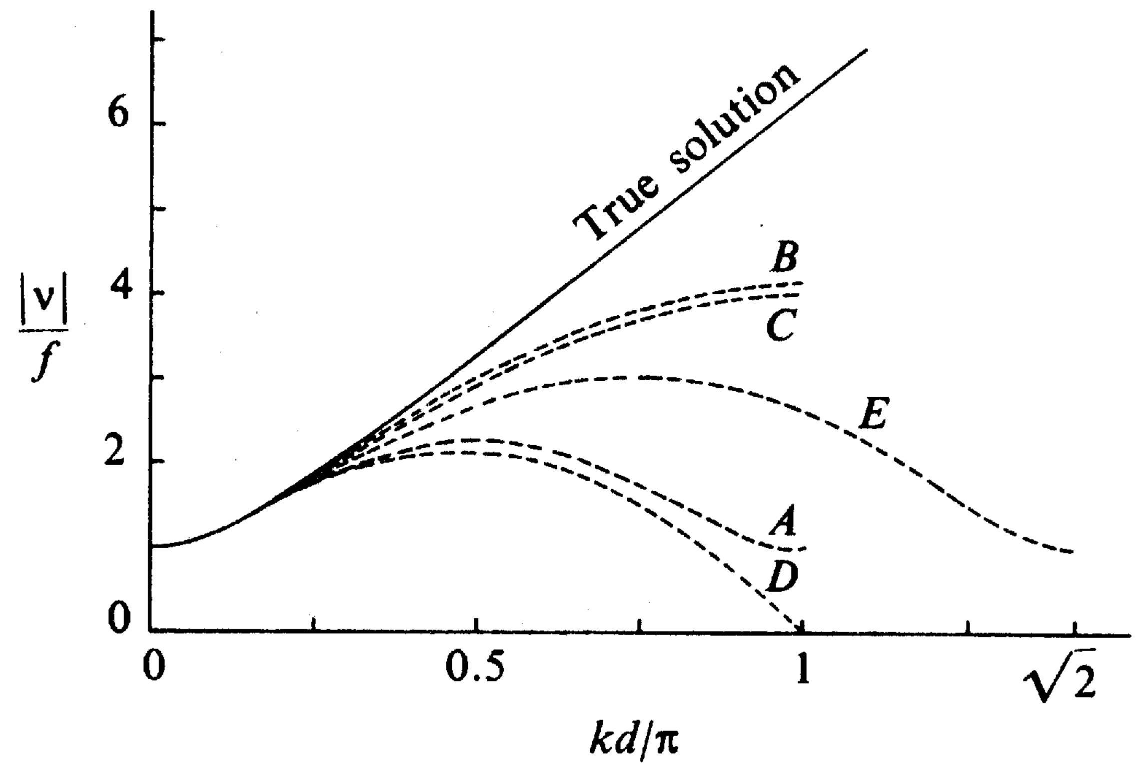

Figure 3.2 **The functions $|\nu|/f$ given by (3.4) and (3.6), with $\lambda/d = 2$.**

The phase speed and group velocities for each of these grids can be plotted together with the exact solution:

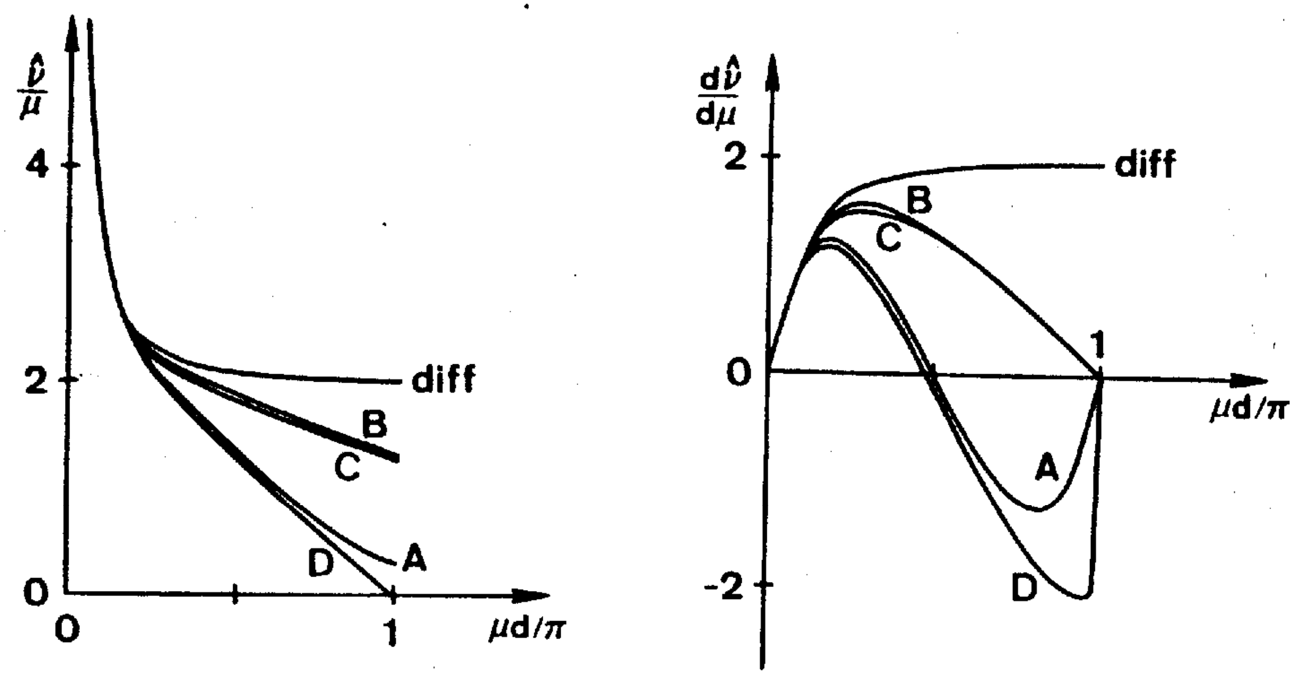

Figure 7.5 The phase velocity $c = \hat{\upsilon}/\mu$ and the group velocity $d\hat{\upsilon}/d\mu$ from Schoenstadt (1978) as functions of $\mu d/\pi$ for the four grids as indicated. The differential solution is also included. These results use the values: $gH = 10^4$ m^2 sec^{-2}, $f = 10^{-4}$ sec^{-1}, $d = 500$ km.

We can see that for the 1-D problem, B and C grid perform the best.

A and D are not good at all. Energy of waves shorter than $4\Delta x$ propagates in the wrong direction.

E is reasonable good.

For 2-D problem, the ω / f is plotted in the following:

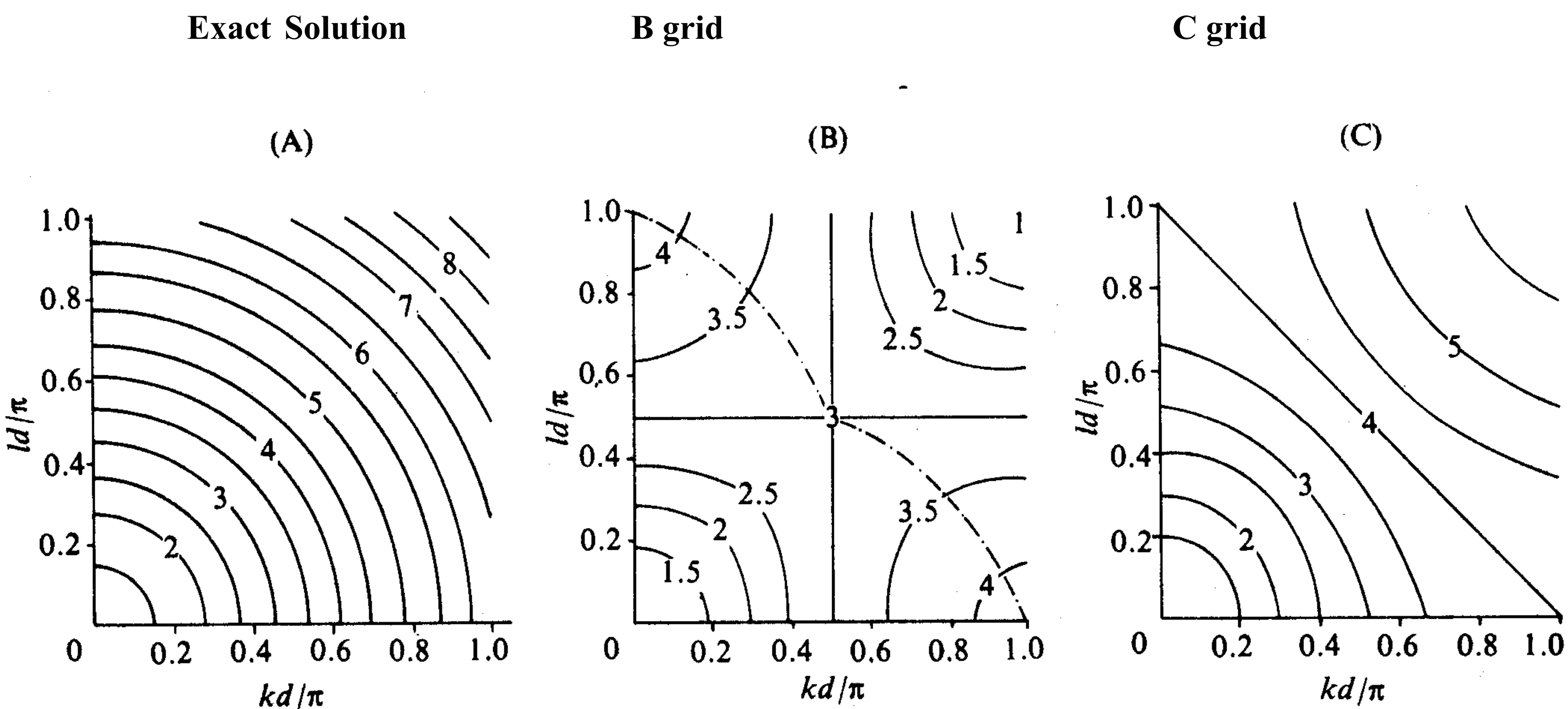

Figure 3.3 The functions $|\nu|/f$, for the true solution and for solutions of systems $(3.2)_B$ and $(3.2)_C$, with $\lambda/d = 2$.

We can see C grid is closest to the exact solution given in (A), and B grid is not as good in 2-D, especially along the diagonal direction in the plot.