V. Controlling nonlinear growth by conservation

A. The conservation method aims at controlling growth by controlling the total energy or other total quadratic properties in an integration domain; this being guided by differential properties.

B. Conservation of the system

$$A_t = -[uA_x + vA_y] = -[(uA)_x + (vA)_y] \qquad (5.19a)$$

$$u_x + v_y = 0 \qquad (5.19b)$$

1. Advective and flux form

a. The first bracketed term in (5.19a) is in advective form.

1'. Eqn. (5.19a) could be written $dA/dt = 0$.

b. The second bracketed term in (5.19a) is in flux form.

1'. It follows from the first bracketed term using (5.19).

2. Differential properties for periodic domain.

a. A is globally conserved,

$$\frac{\partial}{\partial t} \int_D A dxdy = -\int_D [(uA)_x + (vA)_y] dxdy = 0 \qquad (5.20)$$

b. A^2 is globally conserved (quadratic conservation)

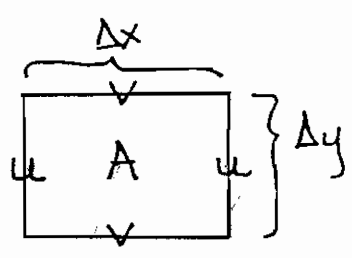
$$\frac{\partial}{\partial t} \int_D (A^2/2) dxdy = -\int_D [(uA^2/2)_x + (vA^2/2)_y] dxdy = 0 \qquad (5.21)$$

3. Conservation in a periodic domain and the flux scheme.

$$\delta_{2t} A = -\delta_x (u\bar{A}^x) - \delta_y (v\bar{A}^y) \qquad (5.22a)$$

$$\delta_x u + \delta_y v = 0 \qquad (5.22b)$$

a. The grid is staggered and (5.22a,b) are both applied at A locations.



b. <u>A is conserved</u>

1'. $$\sum_{i,j} \delta_{2t} A = -\sum_j \overset{0}{\sum_i \delta_x (u\bar{A}^x)} - \sum_i \overset{0}{\sum_j \delta_y (v\bar{A}^y)} = 0 \qquad (5.23a)$$

2'. Therefore, $$\sum_{i,j} A^{n+1} = \sum_{i,j} A^{n-1} \qquad (5.23b)$$

3'. From (5.23b) A is conserved for even and odd time steps.

c. <u>A^2 is semi-conserved</u> (i.e., space derivation terms associated with A^2 behave like those in the differential system).

1'. $$A\delta_{2t}A = -A\delta_x(u\bar{A}^x) - A\delta_y(v\bar{A}^y)$$

$$= -A\{[u_+(A_{++}+A) - u_-(A+A_{--})]/\Delta x$$

$$+ [v^+(A^{++}+A) - v^-(A+A^{--})]/\Delta y\}$$

$$= -(u_+AA_{++}-u_-AA_{--})/\Delta x - (v^+AA^{++}-v^-AA^{--}/\Delta y$$

$$-A^2[(u_+-u_-)\Delta x + (v^+-v^-)/\Delta y] \qquad (5.24a)$$

where $A^+ = A_{i,j+1/2}$, $A_{++} = A_{i+1,j}$, etc., and (5.19b) is used.

2'. $$\sum_i (u_+AA_{++}-u_-AA_{--}) = 0$$

3'. $$\sum_j (v^+AA^{++}-v^-AA^{--}) = 0$$

4'. Therefore, $$\sum_{i,j} A\delta_{2t}A = 0 \quad \text{or} \quad \sum_{i,j} A^nA^{n+1} = \sum_{i,j} A^nA^{n-1} \qquad (5.24b)$$

5'. A^2 is <u>semi-conserved</u>, i.e., A^2 is not conserved (5.24b), but the space terms are (5.24a).

d. Note that $A^n A^{n-1}$ in (5.24b) can be negative or positive at any point so that a solution can grow but still maintain (5.24b). This growth is usually accompanied in this case by a high frequency oscillation in time that results from time differencing.

1'. This can be controlled by a small amount of damping.

e. Aliasing still occurs in quatratically conservative and semi-conservative finite difference systems, but explosive growth due to spacial aliasing can be controlled.

f. If a scheme is only quadratically semi-conservative, the local stability criterion should be satisfied.

1'. Consider the case when u and v are constant [or $u(y)$, $v(x)$], a possible occurrence in a region of a nonlinear problem.

2'. Local stability requires

$$\Delta t \leq \frac{1}{\frac{|u|}{\Delta x} + \frac{|v|}{\Delta y}} \tag{5.25}$$

3'. A forward time scheme should not be used since local growth under these circumstances cannot be prevented.

g. Eqn. (5.22a) can be changed so that it is quadratically conservative, i.e.,

$$\delta_{2t} A = -\delta_x\left(\overline{u\bar{A}^x}^{2t}\right) - \delta_y\left(\overline{v\bar{A}^y}^{2t}\right) \tag{5.26}$$

1'. Local stability occurs for all Δt!

C. Simple two-dimensional incompressible (barotropic) flow in a closed domain.

1. A simple barotropic model

$$\eta_t = -u\eta_x - v\eta_y = \psi_y\eta_x - \psi_x\eta_y = J(\eta,\psi) \qquad (5.27a)$$

$$\eta = \nabla^2\psi = \psi_{xx} + \psi_{yy} = v_x - u_y \qquad (5.27b)$$

$$u = -\psi_y \; , \; v = \psi_x \qquad (5.27c)$$

a. η: vorticity, ψ: stream function, J: Jacobian

2. Global properties in closed domain (no net transport through boundaries) are

$$\frac{\partial}{\partial t} \iint \eta dxdy = \iint J(\eta,\psi) = 0 \qquad (5.28)$$

$$\frac{\partial}{\partial t} \iint \left(\frac{\eta^2}{2}\right) dxdy = \iint \eta J(\eta,\psi) = 0 \qquad (5.29)$$

$$K_t = \frac{\partial}{\partial t} \iint K^* dxdy = -\iint \psi J(\eta,\psi) = 0 \qquad (5.30)$$

a. Conservative properties are η , η^2 and $K^* = (u^2 + v^2)/2$.

b. In a closed domain u = 0 on x boundaries and v = 0 on y boundaries or the boundaries are periodic.

c. Eqn. (5.30) follows from noting that

$$\begin{aligned}
\int \psi\eta_t &= \int \psi(\nabla^2\psi)_t \\
&= \int(\psi\psi_{xt})_x + \int(\psi\psi_{yt})_y - \int\psi_x\psi_{xt} - \int\psi_y\psi_{yt} \\
&= 0 + 0 - \int vv_t - \int(-u)(-u)_t \\
&= -\int\left(\frac{u^2+v^2}{2}\right)_t = -\int K_t^* = -K_t
\end{aligned}$$

3. Series of orthogonal functions

a. The stream function, ψ, can usually be expressed as a series of orthogonal functions:

$$\psi = \sum_\eta \psi_\eta \qquad (5.31)$$

1'. The ψ_n satisfy the Helmholtz eqn.

$$\nabla^2\psi_n + \lambda_n^2\psi_n = 0 \tag{5.32}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

2'. λ_n are the so-called generalized wave numbers.

3'. A simple example is $\psi_n = \sin\lambda_n x$.

4'. Another example is

$$\psi = \sum_{n_1,n_2} [a_{n_1,n_2}\cos(\frac{2\pi n_1}{L_x} x) + b_{n_1,n_2}\sin(\frac{2\pi n_1}{L_x} x)]\sin(\frac{\pi n_2}{L_y} y) \tag{5.33}$$

where ψ is periodic in x, zero at y boundaries and

$$\lambda_n^2 = (\frac{2\pi n_1}{L_x})^2 + (\frac{\pi n_2}{L_y})^2 \tag{5.34}$$

b. If ψ_n satisfy (5.32) they are orthogonal, i.e.

$$\iint\psi_n\psi_m = 0 \text{ if } n \neq m \tag{5.35}$$

1'. $\psi_m[\nabla^2\psi_n + k_n^2\psi_n] = 0$

$\psi_n[\nabla^2\psi_m + k_m^2\psi_m] = 0$

2'. $\iint[\psi_m\nabla^2\psi_n - \psi_n\nabla^2\psi_m + (k_n^2-k_m^2)\psi_n\psi_m] = 0$

3'. $$\iint[\nabla\cdot(\psi_m\nabla\psi_n) - \nabla\psi_m\nabla\psi_n - \nabla\cdot(\psi_n\nabla\psi_m) + \nabla\psi_n\nabla\psi_m + (k_n^2-k_m^2)\psi_n\psi_m] = (k_n^2-k_m^2)\iint\psi_n\psi_m = 0 \tag{5.36}$$

provided

$$\iint[\nabla\cdot(\psi_m\nabla\psi_n) - \nabla\cdot(\psi_n\nabla\psi_m)] = 0 ,$$

i.e., no net transport of mass through domain (boundaries periodic or normal velocities zero).

4'. (5.36) holds if $n \neq m$ if $\iint\psi_n\psi_m = 0$

4. Average kinetic energy

a. Define the average kinetic energy as

$$\overline{K} = \frac{1}{2}\overline{(u^2+v^2)} = \frac{1}{2}\overline{\nabla\psi\cdot\nabla\psi} \qquad (5.37)$$

where

$$\overline{\alpha} = \frac{1}{A}\int_A \alpha dA$$

b. Using (5.31) and again assuming no net mass transport through boundaries

$$\overline{K} = \frac{1}{2}\overline{\nabla\sum_n\psi_n\cdot\nabla\sum_m\psi_m}$$

$$= \frac{1}{2}\overline{\sum_n\nabla\psi_n\cdot\sum_m\nabla\psi_m}$$

$$= \frac{1}{2}\sum_n\sum_m\overline{\nabla\psi_n\cdot\nabla\psi_m}$$

$$= \frac{1}{2}\sum_n\sum_m\overline{\nabla\cdot(\psi_m\nabla\psi_n) - \psi_m\nabla^2\psi_n} \qquad \text{(no net transport)}$$

$$= -\frac{1}{2}\sum_n\sum_m\overline{\psi_m\nabla^2\psi_n} \qquad \text{(from (5.32))}$$

$$= \frac{1}{2}\sum_n\sum_m\lambda_n^2\,\overline{\psi_m\psi_n} \qquad \text{(orthogonality)}$$

$$= \frac{1}{2}\sum_n\lambda_n^2\,\overline{\psi_n^2}$$

c. Thus

$$\overline{K} = \sum_n K_n \qquad (5.38)$$

where $K_n = \frac{1}{2}\lambda_n^2\,\overline{\psi_n^2}$ represents the kinetic energy of wave number λ_n.

5. Average square vorticity

a. Define the average square vorticity as

$$\overline{\eta^2} = \overline{(\nabla^2\psi)^2} \tag{5.39}$$

b. Substituting (5.31) into (5.39), assuming no net mass transport through boundaries and using (5.38)

$$\overline{\eta^2} = \sum_n \lambda_n^4 \overline{\psi_n^2} = 2\Sigma\lambda_n^2 K_n \tag{5.40}$$

1'. Enstrophy is $\overline{\eta^2}/2$

6. Average wave number, λ.

a. Definition:

$$\lambda^2 = \sum_n \lambda_n^2 K_n / \sum_n K_n = \overline{\eta^2}/2\overline{K} \tag{5.41}$$

b. (5.41) is a ratio of the average enstrophy and kinetic energy.

7. Energy flow is restricted.

a. The mean wave number, λ, is constant.

1'. From (5.29) and (5.30) $\overline{K}$ and $\overline{\eta^2}$ are conserved in time.

2'. Thus, λ is also from (5.41).

b. Consider an example with 3 wave numbers satisfying $\lambda_1^2 > \lambda_2^2 > \lambda_3^2$.

1'. From (5.38) and (5.40)

$$K_1 + K_2 + K_3 = \overline{K} > 0 \tag{5.42a}$$

$$\lambda_1^2 K_1 + \lambda_2^2 K_2 + \lambda_3^2 K_3 = \overline{\eta}^2/2 > 0 \tag{5.42b}$$

2'. Eliminating K_1 from (5.42a) and K_3 from (5.42b)

$$(\lambda_1^2-\lambda_2^2)K_2 + (\lambda_1^2-\lambda_3^2)K_3 = \lambda_1^2\overline{K} - \overline{\eta}^2/2 > 0 \tag{5.43a}$$

$$(\lambda_1^2-\lambda_3^2)K_1 + (\lambda_2^2-\lambda_3^2)K_2 = \overline{\eta}^2/2 - \lambda_3^2\overline{K} > 0 \tag{5.43b}$$

3'. $\lambda_1^2 C_1 - C_2$ and $C_2 - \lambda_3^2 C_1$ are greater than zero since the left sides of (5.43) are for K_1, K_2, K_3 varying.

4'. If (5.43) is to hold when K_2 increases, K_1 and K_3 must decrease and when K_2 decreases, K_1 and K_3 must increase or energy transfers follow

$$K_1 \rightarrow K_2 \leftarrow K_3 \tag{5.44a}$$

$$K_1 \leftarrow K_2 \rightarrow K_3 \tag{5.44b}$$

5'. Further, there is a maximum transfer to K_2 in (5.44a) (i.e., when $K_1 = 0$ or $K_3 = 0$) and to K_1 and K_3 in (5.44b) (i.e., when $K_2 = 0$).

8. Finite difference methods for (5.27) that semi-conserve kinetic energy and enstrophy

a. Semi-conservation of kinetic energy and enstrophy require that the finite difference approximations to the Jacobian in (5.27a), J_ℓ, satisfy respectively

$$\sum_{i,j} \psi J_\ell(\eta,\psi) = 0 \tag{5.45a}$$

$$\sum_{i,j} \eta J_\ell(\eta,\psi) = 0 \tag{5.45b}$$

b. Three forms of the Jacobian and their finite difference counterparts

1'. $$J(\eta,\psi) = \eta_x\psi_y - \eta_y\psi_x \tag{5.46a}$$

$$J_1 = (\delta_{2x}\eta)(\delta_{2y}\psi) - (\delta_{2y}\eta)(\delta_{2x}\psi) \tag{5.46b}$$

$$J(\eta,\psi) = (\psi\eta_x)_y - (\psi\eta_y)_x \tag{5.47a}$$

$$J_2 = \delta_{2y}(\psi\delta_{2x}\eta) - \delta_{2x}(\psi\delta_{2y}\eta) \tag{5.47b}$$

$$J(\eta,\psi) = (\eta\psi_y)_x - (\eta\psi_x)_y \tag{5.48a}$$

$$J_3 = \delta_{2x}(\eta\delta_{2y}\psi) - \delta_{2y}(\eta\delta_{2x}\psi) \tag{5.48b}$$

2'. In all cases

$$\eta = \delta_{xx}\psi + \delta_{yy}\psi \qquad (5.49)$$

c. Semi-conservative schemes and practical implications (Arakawa and Lamb, 1977: The UCLA General Circulation Model, Methods in Computational Physics).

1'. $\zeta = \eta$, $d = \Delta x = \Delta y$ in the following.

2'. Note J_7 helps preserve mean wave number.

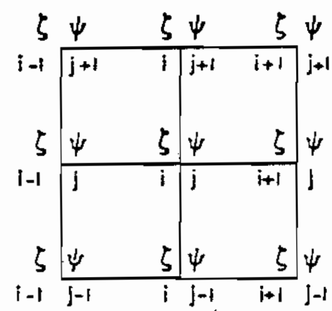


FIG. 10. Grid showing indexing for ζ, ψ points used in the finite-difference Jacobian schemes of Eq. (58).

It was shown by Arakawa (1966) that the Jacobian J given by

$$J = \alpha J_1 + \gamma J_2 + \beta J_3 , \quad \alpha + \gamma + \beta = 1 , \qquad (59)$$

conserves mean square vorticity if $\alpha = \beta$ and conserves energy if $\alpha = \gamma$. Examples of Jacobians which have the form of (59) are

$$J_4 = \frac{1}{2}(J_1 + J_2) ,$$

$$J_5 = \frac{1}{2}(J_2 + J_3) ,$$

$$J_6 = \frac{1}{2}(J_3 + J_1) ,$$

$$J_7 = \frac{1}{3}(J_1 + J_2 + J_3) . \qquad (60)$$

A schematic representation of the ζ and ψ points used in constructing the seven finite-difference Jacobians introduced above is given in Fig. 11.

J_7 is the Jacobian proposed by Arakawa (1966) as conserving both enstrophy and energy. J_2 and J_6 conserve enstrophy, but not energy. J_3 and J_4 conserve energy, but not enstrophy. All five schemes mentioned thus far are stable. J_1 does not conserve either quantity. J_5, also, does not conserve either quantity, but experience with numerical tests shows that the instability is very weak, if it exists at all. This is not surprising, since $2J_5 = 3J_7 - J_1$; because J_7 is a quadratic-conserving scheme the time rates of change of the mean quadratic quantities using J_5, for given ζ and ψ, have opposite sign to the time rates of change of the mean quadratic quantities using J_1.

AKIO ARAKAWA AND VIVIAN R. LAMB

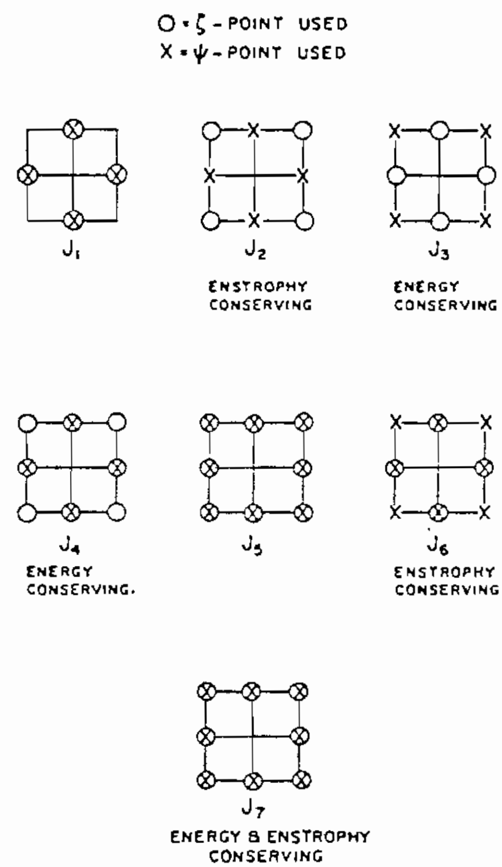


FIG. 11. Schematic representation of ζ and ψ points used in constructing the finite-difference Jacobians defined by Eqs. (58) and (60).

J_7 is the best second-order scheme because of its formal guarantee for maintaining the integral constraints on the quadratic quantities. J_7 is also just as accurate as any other second-order scheme. A further increase in accuracy can be obtained by going to higher order schemes. The more accurate fourth-order scheme that has the same integral constraints as J_7 was also given by Arakawa (1966).

Numerical tests have been made with the above seven Jacobians. In these tests, the initial condition was given by

$$\psi = \Psi \sin(\pi i/8)[\cos(\pi j/8) + 0.1 \cos(\pi j/4)], \quad (61)$$

and Δt was chosen such that $\Delta t/d^2 = 0.7$. The leapfrog scheme was used instead of the implicit scheme. In order to eliminate the gradual separation of the solutions at even and odd time steps that occurs in the leapfrog scheme, a two-level scheme was inserted every 240 time steps. The simplest five-point Laplacian was used. Figures 12 and 13 show the time change of enstrophy and energy obtained with the seven Jacobians. The expected conservation properties are observed, even though the implicit scheme was not used. The energy conserving schemes J_3 and J_4 show considerable increase of enstrophy. On the other hand, the enstrophy conserving schemes J_2 and J_6 approximately conserve energy in spite of the lack of a formal guarantee. This is reasonable

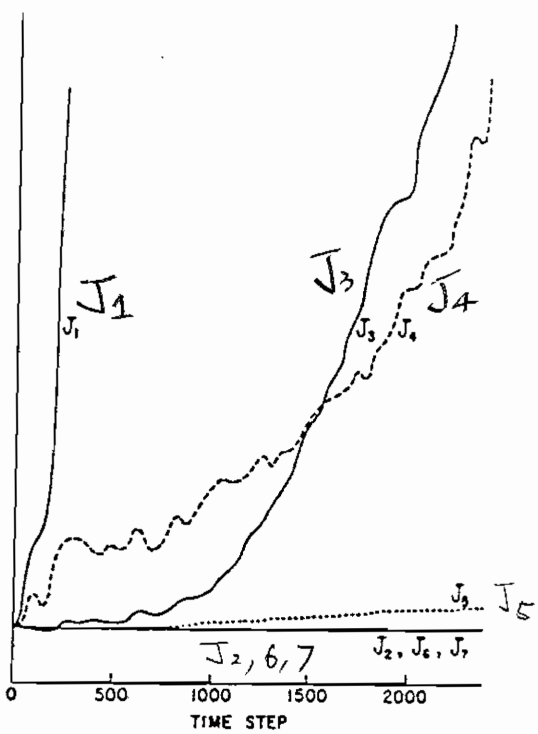


FIG. 12. Comparison of the time variation of the mean square vorticity (units arbitrary) during a numerical integration with the seven finite-difference Jacobians under consideration. (Arakawa, 1970). Reprinted with permission of the publisher American Mathematical Society from *SIAM-AMS Proceedings*. Copyright © 1970, Vol. 2, Fig. 5, p. 35.

because the enstrophy is more sensitive to shorter waves for which the truncation errors are large. J_5 approximately conserves both quantities, again in spite of the lack of formal guarantees. J_7 conserves both quantities

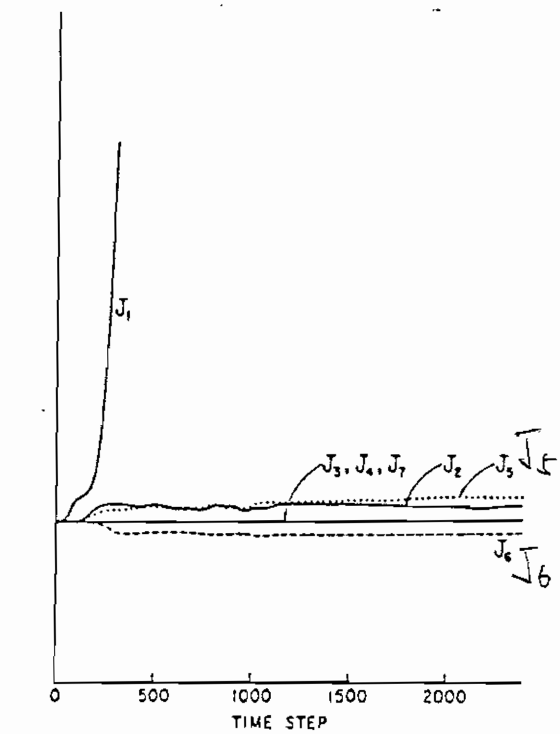


FIG. 13. Comparison of the time variation of the kinetic energy during a numerical integration with the seven finite-difference Jacobians under consideration (Arakawa, 1970). Reprinted with permission of the publisher American Mathematical Society from *SIAM-AMS Proceedings*. Copyright © 1970, Vol. 2, Fig. 6, p. 36.

with only negligible errors arising from the leapfrog scheme. J_5, like J_1 and J_7, maintains the property of the Jacobian $J(\zeta,\psi) = -J(\psi,\zeta)$.

Figure 14 shows the spectral distribution of kinetic energy obtained by the energy and enstrophy conserving scheme J_7 and by the energy conserving scheme J_3 at the end of the calculations. The small arrow shows the wave number for sin $(\pi i/8)$ cos $(\pi j/8)$, which contained almost all of the energy at the initial time. Although the total energy was approximately conserved with J_3 there was a considerable spurious energy cascade into the high wave numbers, whereas with J_7 more energy went into a lower wave number than into the higher wave numbers, in agreement with the conservation of the average wave number as given by Eq. (52).

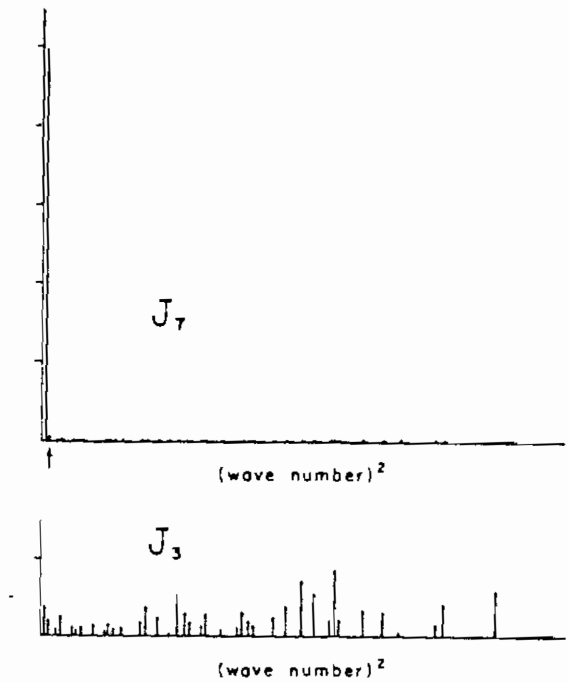


FIG. 14. A comparison of the spectral distribution of kinetic energy, obtained with J_3 and J_7, after a numerical integration of 2400 time steps. Arrow shows the wave number that contained most of the energy at the initial time.

Whether the increase of the enstrophy is important in the simulation of large-scale atmospheric motion will depend on the viscosity used with the complete equation. A relatively small amount of viscosity may be sufficient to keep the enstrophy quasi-constant in time. However, the viscosity will also remove energy, and as a result the average wave number, defined by Eq. (52), will falsely increase with time.

In Section II it was pointed out that when a scheme that produces a strong computational cascade is used, a decrease in grid size does not mean an increase in overall accuracy as far as long-term numerical integrations are concerned. Figure 15 shows such an example. With an identical initial condition, experiments have been made using J_3 with three different grid sizes. The nondimensional parameter $\Psi\Delta t/d^2$ is kept the same for the three experiments. A two-level scheme was inserted every 120 time steps to suppress separation of the solution due to the leapfrog scheme. The figure shows a more rapid increase of enstrophy with the smaller grid sizes. Since the kinetic energy is practically conserved in all three experiments, a larger enstrophy means a smaller average scale of the motion. These results show that the convergence of the scheme, in the nonlinear sense, must be seriously questioned.

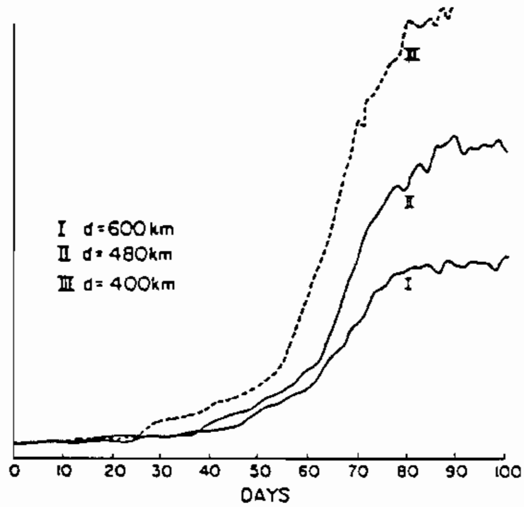


FIG. 15. A comparison of the time variation of mean square vorticity obtained by numerical integrations using J_3 for three different grid sizes.

d. The effect of time differencing on $\eta^2/2$ conservation is

$$\frac{1}{2}\left[\overline{(\eta^{n+1})^2} - \overline{(\eta^n)^2}\right] = \overline{\{[(\eta^{n+1}+\eta^n)/2]-\eta^*\}(\eta^{n+1}-\eta^n)}$$

$$+ \overline{\Delta t\ \eta\ J^*_\ell(\eta^*,\psi^*)} \qquad (5.50)$$

where

$$\eta^{n+1} - \eta^n = \Delta t\ J_\ell(\eta^*,\psi^*) \qquad (5.51)$$

1'. (5.50) follows from (5.51) after multiplication of (5.50) by $(\eta^{n+1}+\eta^n)/2$.

2'. If $\eta^*\ J_\ell(\eta^*,\psi^*) = 0$ the average change in η^2 is governed by the first term on the right of (5.50).

3'. Leapfrog with time step $\Delta t/2$ when $\psi^* = \psi^{n+1/2}$, $\eta^* = \eta^{n+1/2}$.

4'. Crank-Nicholson when $\psi^* = (\psi^{n+1}+\psi^n)/2$ and $\eta^* = (\eta^{n+1}+\eta^n)/2$.

5'. Note that when $\eta^* = (\eta^{n+1}+\eta^n)/2$

$$(\eta^{n+1})^2 = (\eta^n)^2$$

6'. Matsuno when $\psi^* = \psi^n$, $\eta^* = \eta^n + \frac{\Delta t}{2} J(\eta^n,\psi^n)$.