Quiz #3. Physical Mechanics, 2000
Total 9 points.

Answers

1. (3 points) Determine which of the following is scalar and which is vector?

Gradient of a scalar field \( \nabla \phi \) is a _____Vector_____________
Divergence of a flow (velocity) field \( \nabla \cdot \vec{V} \) is a _____Scalar_________
Vorticity \( \nabla \times \vec{V} \) is a _____Vector_________

2. (3 points) For a particle undergoing a uniform circular motion, the position vector of the particle is \( \vec{r} = \hat{i} r \cos(\omega t) + \hat{j} r \sin(\omega t) \). Here \( r \) is the radius of the circle and \( \omega \) the angular velocity, both are constant.

a. Find the velocity (vector) for the particle at time \( t \).

\[
\vec{V} = \frac{d\vec{r}}{dt} = -\hat{i}\omega r \sin(\omega t) + \hat{j}r\omega \cos(\omega t)
\]

b. Show (from their definitions) that the velocity is always perpendicular to the position vector.

When \( \vec{V} \cdot \vec{r} = 0 \), \( \vec{V} \perp \vec{r} \).

\[
\vec{V} \cdot \vec{r} = \vec{V} \cdot [\hat{i} r \cos(\omega t) + \hat{j} r \sin(\omega t)] = -\omega r \sin(\omega t) r \cos(\omega t) + r \omega \cos(\omega t) r \sin(\omega t) = 0
\]
Therefore \( \vec{V} \perp \vec{r} \).

c. What does it say about the direction of motion of this particle? Use diagram if you want.

The particle moves along a circle of radius \( r \), and direction is tangential to the circle.

3. (3 points) If force \( \vec{F} \) can be written in terms the gradient of scalar \( \phi \), i.e.,

\[
\vec{F} = \nabla \phi
\]

show (yes, prove) that the work done by this force along any closed path is always zero, i.e., \( W = \oint \vec{F} \cdot d\vec{r} = 0 \). (Hint: First find the work done by this force alone a path starting at \( P_1 \) and ending at \( P_2 \), then show that if \( P_1 \) and \( P_2 \) are the same point, i.e., if
the path is closed, the work is zero. You may need to use the definition of total
differential \( d\phi = \frac{\partial \phi}{\partial x} \, dx + \frac{\partial \phi}{\partial y} \, dy + \frac{\partial \phi}{\partial z} \, dz \). Since
\[
\vec{F} = \nabla \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} \quad \text{and} \quad d\vec{r} = \hat{i} \, dx + \hat{j} \, dy + \hat{k} \, dz
\]
\[
\vec{F} \cdot d\vec{r} = \frac{\partial \phi}{\partial x} \, dx + \frac{\partial \phi}{\partial y} \, dy + \frac{\partial \phi}{\partial z} \, dz = d\phi \rightarrow
\]
\[
W = \oint\vec{F} \cdot d\vec{r} = \oint d\phi = 0.
\]
Or use the Stokes Theorem:
\[
W = \oint \vec{F} \cdot d\vec{r} = \iint (\nabla \times \vec{F}) \cdot \hat{n} \, ds = \iiint (\nabla \times \nabla \phi) \cdot \hat{n} \, ds \quad \text{because}
\]
\[
\nabla \times \nabla \phi = \left|\begin{array}{ccc}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z}
\end{array}\right| = \hat{i} (\phi_{yz} - \phi_{zy}) - \hat{j} (\phi_{zx} - \phi_{xz}) + \hat{k} (\phi_{xy} - \phi_{yx}) = 0
\]