Mesoscale
Meteorological
Modeling

Roger A. Pielke
Department of Atmospheric Science
Colorado State University
Fort Collins, Colorado

1984
When \( \Delta x^2 > 4 \), \( \cos \alpha \Delta x = 0 \), so that the imaginary component can be written as

\[
\Delta x = -\frac{1}{2} \left( \sqrt{4 - \Delta x^2} \right).
\]

To ascertain whether this quantity is less than or greater than unity, let

\[
\Delta x = 2 + \epsilon
\]

where \( \epsilon > 0 \), so that

\[
\Delta x = -\frac{1}{2} \left( \sqrt{4 - \Delta x^2} + \epsilon^2 \right).
\]

Since either root is possible

\[
\Delta x = -1 - \frac{1}{2} \frac{\Delta x - 1}{\sqrt{4 - \Delta x^2} + \epsilon^2}
\]

\( (\Delta x) > 1 \) so that when \( \Delta x^2 > 4 \), the leapfrog scheme is linearly unstable. Since

\[
\Delta x^2 = 4 \Delta x \sin^2 \Delta x
\]

stability is retained only when

\[
C \sin \Delta x \Delta x \leq 1
\]

or since the maximum value of \( \sin \Delta x \Delta x \) is unity for a \( 4 \Delta x \) wave \( k = n/2 \), then

\[
|C| < 1
\]

is a necessary and sufficient condition for the linear stability of the scheme.

The ratio of the predicted phase speed to the advection velocity for this technique can be obtained by dividing the imaginary by the real component for \( \Delta x^2 > 4 \) and solving for the phase speed. Since \( \Delta x = 1 \), however, it is not possible to use either the imaginary or real components separately to obtain the phase speed. Using the imaginary part, therefore, gives the rate of the change in phase speed to

\[
\frac{\partial \Theta}{\partial x} = \frac{1}{\Delta x} \sin \left( \frac{1}{2} \frac{\Delta x}{\Delta x} \right)
\]

Because of the quadratic form of (10-18), two wave solutions occur. One moves downstream \( (\Delta x^2 > 0 \text{ when } C > 0) \) and is related to the real solution of the advection equation, and the other moves upstream and is called the computational mode. The computational mode occurs because the leapfrog is a second-order difference equation. Such a separation of solutions by the centered-in-time, leapfrog scheme can be controlled by occasionally averaging in time to assure that the even and odd time steps remain consistent with one another. As long as the time steps are consistent the amplitude of the computational mode is small.

Values of \( \lambda \) and \( \Delta x \) for different values of \( C \) and wavelength are displayed in Table 10-1. Although the leapfrog scheme preserves amplitudes exactly

as long as \( |C| \leq 1 \), the accuracy of the phase representation deteriorates markedly for the shorter wavelengths. Because the numerical representation of these waves travels more slowly than the true solution, the scheme is said to be dispersive since waves of different wavelengths are necessarily superimposed, they will travel with different speeds relative to one another even if the advection velocity is a constant. The retention of these dispersive shorter waves in the solution can cause computational problems through nonlinear instability, as discussed in Section 10.6.5. The important conclusions obtained from the analysis of the leapfrog scheme is that the exact representation of the amplitude does not by itself guarantee success in simulations since the fictitious dispersion of waves of different lengths can generate errors. Baer and Simons (1970), for example, have reported that in approximating nonlinear advection terms, individual energy components may have large errors when the total energy has essentially none. They conclude that neither conservation of integral properties nor satisfactory prediction of amplitude is sufficient to justify confidence in the results—one must also assure the accurate calculation of phase speed.

In both the forward and upstream leapfrog schemes that we have examined, the time step must be less than or equal to the time it takes a 4

change or grid point to be translated by advection to the next grid point downstream. When we generalize this result to all types of wave propagations, we need to filter rapidly moving waves, which are not considered important on the mesoscale, as apparent. This is the reason that scale analysis is used to derive simplified conservation relations [e.g., the anelastic conservation of mass equation (8-11)] so that sound waves can be eliminated as a possible solution, as shown in Section 5.2.2.
The exact solution to the diffusion equation (the left-hand side of (10-19)) with \( K \) equal to a constant, i.e., \( \frac{\partial}{\partial t} K = K \frac{\partial^2}{\partial x^2} \), can be determined by assuming

\[
\overline{\phi} = \phi_{00} e^{\alpha x} e^{\beta y},
\]

where no damping in the z-direction is permitted (i.e., \( K_z = 0 \)). Substituting this expression into the linearized diffusion equation and simplifying, yields

\[
\alpha_0 - \alpha_n = -K_k^2
\]

where the subscript \( r \) on \( k \) has been eliminated to simplify the notation. Equating real and imaginary components shows that \( \alpha_0 = 0 \) so that the exact solution can be written as

\[
\overline{\phi} = \phi_{00} e^{\alpha x} e^{\beta y}.
\]

Expressing the dependent variables as a function of frequency and wavenumber, (10-20) can be rewritten as

\[
\phi_1 = 1 + y \left( \phi_1 - 2 + \phi_{-1} \right) = 1 + 2y \cos k \Delta z - 1,
\]

where \( y = K \Delta x/(2 \Delta z) \) and \( \phi_1, \phi_{-1} = 2 \cos k \Delta z \). The nondimensional parameter \( y \) is called the Fourier number. Equating real and imaginary components yields

\[
\Delta z \cos \alpha_n = 2 y \cos k \Delta z - 1,
\]

\[
\sin \alpha_n = 0.
\]

Since \( \sin \alpha_n, \Delta z \) must be identically equal to zero, \( \alpha_n, \Delta z \) and, therefore, the phase speed are also equal to zero. Thus the solution to (10-20) does not propagate as a wave but amplitudes or decays in place. Since \( \cos \alpha_n, \Delta z = 1 \), the real part can be divided by the analytic solution, \( \phi_0 = e^{-\alpha x} = e^{-\alpha z_0} \alpha_n \) and rewritten as

\[
\phi_1 = 1 + 2y \cos k \Delta z - 1,
\]

where \( n \) is the number of grid points per wavelength. For very long waves (\( n \approx z_0 \)) \( \phi_1 = 1 \) and \( \Delta z = 1 \) since \( \cos \Delta z = (2 \zeta) \Delta z = 1 \), and, therefore, no damping or amplification occurs. For the shortest waves that can be resolved \( L \approx 2 \Delta z, n \approx 2 \),

\[
\alpha_1 = 1 - 4y.
\]

To assure that the magnitude of \( \alpha_1 \) is less than unity and, therefore, computationally stable, \( 4y \) must be less than or equal to 2 or

\[
y \leq 1.
\]

The condition \( y \leq 1 \), however, causes \( \alpha_1 \) to switch between +1 and -1 each

\[
y \approx 1.
\]
where, as with the explicit scheme, \( \gamma = K \Delta t / (\Delta x)^2 \). Since

\[ \dot{\phi} = \psi \psi' \quad \text{and} \quad \phi' = \psi' \psi, \]

\[ \psi' = 1 + \gamma [\psi (\psi - 2 + \psi') + \psi' (\psi - 2 + \psi')] \]

or

\[ \psi' = \frac{1 + \gamma (\psi - 2 + \psi')}{1 - \gamma (\psi - 2 + \psi')} = \frac{1 + \gamma \psi (\Delta x)^2}{1 - \gamma \psi (\Delta x)^2} \]

where as with the analysis of the explicit representation, the imaginary part is zero so that \( \lambda = \psi \).

Values of the ratio of the computational approximation of the damping to the analytic damping \( \lambda/\lambda_p \) are presented in Table 10.2 as a function of wavelength and \( \gamma \). For a given value of \( \gamma \), the 2\(\Delta x \) wave is more poorly approximated. In addition, the 2\(\Delta x \) wave is always sufficiently damped and often the value of \( \lambda \) is negative, yielding a wave whose amplitude reverses (flip-flops) each time step. The solutions become more accurate as \( \gamma \) becomes smaller, and the implicit representation gives reasonable results for large wavelengths even when the explicit form is linearly unstable for all spatial scales.

Equation (10-21) can be written in the following form

\[ \frac{\Delta t \dot{\phi}}{\Delta x} = \left[ 1 + \frac{\Delta t K}{\Delta x^2} \right] \phi - \left[ 1 + \frac{\Delta t K}{\Delta x^2} \right] \frac{\Delta K}{\Delta x^2} \phi' \]

and solved for nonperiodic boundary conditions using a procedure described in Section 10.2. Its solution for periodic boundary conditions is given in Appendix A.

When \( \beta = \beta_p \), this representation is called the Crank–Nicholson scheme. Pimple et al. (1976) have presented results that show that \( \beta = 0.75 \) provides a representation as accurate as the explicit scheme but with a much longer permissible time step. Figure 10.3, reproduced from Muller and Pesch (1975a), illustrates predictions of the growth of a heated boundary layer using both the explicit representation of diffusion given by (10-19) and the implicit form (10-21) with \( \beta = 0.75 \). As reported in that paper, use of the implicit form permitted a much longer time step so that this calculation ran 17 times longer than when the explicit form was used.
10.3 Coriolis Terms

The implicit scheme can also be shown to be a necessity for the Coriolis terms. The terms dealing with the rotation of the earth (see (4.20)) are already in linear form and with \( T_s = L_s \), can be written as

\[
\frac{\partial \theta}{\partial t} = f; \quad \frac{\partial \phi}{\partial t} = -fa_f.
\]

(10.22)

If these relations are approximated using an explicit representation, they are written as

\[
\theta^{n+1} - \theta^n / \Delta t = f; \quad \phi^{n+1} - \phi^n / \Delta t = -fa_f.
\]

(10.23)

Requiring the dependent variables in terms of frequency and wavenumber, and rearranging yields

\[
\frac{\partial^2 \phi}{\partial \omega^2} - \frac{\partial \phi}{\partial \omega} = 0,
\]

\[
\frac{\partial^2 \phi}{\partial \sigma^2} + \frac{\partial \phi}{\partial \sigma} = 0,
\]

where \( \phi \) and \( \lambda \) are functions of \( \sigma \) and \( \omega \). In matrix form these equations can be written as

\[
\begin{bmatrix}
\phi^{n+1} - \phi^n / \Delta t & \phi^{n+1} - \phi^n / \Delta t
\end{bmatrix}
\begin{bmatrix}
\Delta f \\
\phi^{n+1} - \phi^n / \Delta t
\end{bmatrix} = \begin{bmatrix}
0
\end{bmatrix}
\]

As shown preceding (5.31), this homogeneous set of algebraic equations has a solution only if the determinant of the coefficients is equal to zero, thus

\[
(\phi^{n+1} - \phi^n / \Delta t)^2 + (\Delta f)^2 = \phi^2 - 2\phi^n + 1 + (\Delta f)^2 = 0.
\]