Quiz #3. Physical Mechanics, 2000
Total 9 points.

Answers

1. (3 points) Determine which of the following is scalar and which is vector?

   Gradient of a scalar field (∇f) is a _______Vector_________________
   Divergence of a flow (velocity) field $\nabla \cdot \vec{V}$ is a _____Scalar_________
   Vorticity $\nabla \times \vec{V}$ is a _____Vector____________

2. (3 points) For a particle undergoing a uniform circular motion, the position vector of the particle is $\vec{r} = \hat{i}\, r \cos(wt) + \hat{j} r \sin(wt)$. Here r is the radius of the circle and ω the angular velocity, both are constant.

   a. Find the velocity (vector) for the particle at time t.

   $$\vec{V} = \frac{d\vec{r}}{dt} = -\hat{i} w\, r \sin(wt) + \hat{j} rw \cos(wt)$$

   b. Show (from their definitions) that the velocity is always perpendicular to the position vector.

   When $\vec{V} \cdot \vec{r} = 0$, $\vec{V} \perp \vec{r}$.
   $\vec{V} \cdot \vec{r} = \vec{V} = [-\hat{i} w\, r \sin(wt) + \hat{j} rw \cos(wt)] \cdot [\hat{i}\, r \cos(wt) + \hat{j} r \sin(wt)]$
   $= -w\, r \sin(wt) r \cos(wt) + rw \cos(wt) r \sin(wt) = 0$
   Therefore $\vec{V} \perp \vec{r}$.

   c. What does it say about the direction of motion of this particle? Use diagram if you want.

   The particle moves along a circle of radius r, and direction is tangential to the circle.

3. (3 points) If force $\vec{F}$ can be written in terms the gradient of scalar f, i.e.,

   $$\vec{F} = \nabla f$$

   show (yes, prove) that the work done by this force along any closed path is always zero, i.e., $W = \oint \vec{F} \cdot d\vec{r} = 0$. (Hint: First find the work done by this force alone a path starting at P_1 and ending at P_2, then show that if P1 and P2 are the same point, i.e., if the path is closed, the work is zero. You may need to use the definition of total differential $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$).

Since

$$\vec{F} = \nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} \text{ and } d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

$$\vec{F}\cdot d\vec{r} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df \rightarrow$$

$$W = \oint \vec{F}\cdot d\vec{r} = \oint df = 0 .$$

Or use the Stokes Theorem:

$$W = \oint \vec{F}\cdot d\vec{r} = \iint (\nabla\times\vec{F})\cdot\vec{n}ds = \iint (\nabla\times\nabla f)\cdot\vec{n}ds \text{ because}$$

$$\nabla\times\nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \hat{i}(f_{zy} - f_{yz}) - \hat{j}(f_{zx} - f_{xz}) + \hat{k}(f_{xy} - f_{yx}) = 0$$