

**Final Comprehensive Exam
Physical Mechanics
Friday December 15, 2000**

**Total 100 Points
Time to complete the test: 120 minutes**

**Please Read the Questions Carefully and Be Sure to Answer All Parts!
In case that you have difficulty getting to the final answer, try at least to put down
the steps of solution with the right equations.**

**Try to use sketches to help you understand the questions whenever possible.
Some questions require only simple answers**

Good Luck!

Answers

1. (10%) State in words the Newton's Second Law of motion and the reference frame in which it is valid.

The Newton's Second Law of motion says that in an inertial reference frame, the **change in the momentum (or the mass time acceleration)** of an object is **equal to the net force** applied to this object.

It's valid only in an **inertial** (non-accelerating) reference frame but can be used if the acceleration of the reference frame is taken into account, in the form apparent forces.

2. (10%) Assume the atmosphere has a constant lapse rate ($g = -\frac{\partial T_{atmosphere}}{\partial z}$) of 5K/km, an air parcel undergoing adiabatic vertical motion has an adiabatic lapse ($\Gamma_d = -\frac{dT_{parcel}}{dz}$) of 9.76K/km.

- (a) (3%) For an air parcel displaced vertically from its initial height level $z=0$, what kind of force will it be subject to?

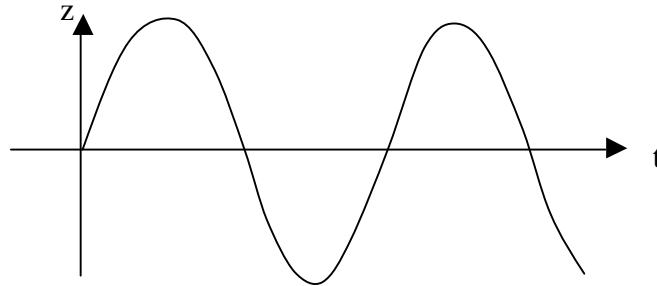
Since the air parcel's temperature decreases with height faster than it's surrounding atmosphere, it becomes cooler (warmer) than its surrounding when displaced upward (downward), therefore it is subjecting to a **restoring** force.

- (b) (3%) What's the stability (stable, neutral or unstable) of the atmosphere?

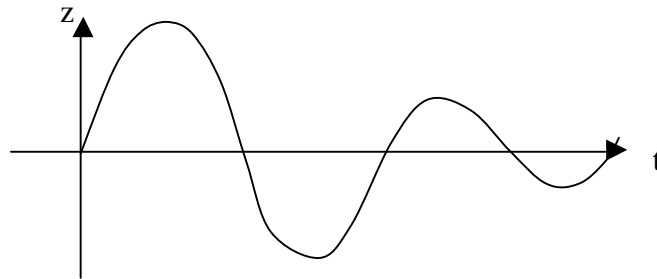
Since the displaced parcel is subjecting to a restoring force, the atmosphere is **stable**.

(c) (4%) Sketch the vertical coordinate, $z(t)$, of this displaced air parcel as a function of time, *with and without friction*.

Without friction, the parcel undergoes simple harmonic oscillations with constant amplitude.



With friction, the parcel undergoes damped harmonic oscillations with the amplitude exponentially decreasing in time.



3. (10%) Give the physical definitions of the terms in equation

$$\frac{\partial F}{\partial t} = \frac{dF}{dt} - \vec{V} \cdot \nabla F .$$

(5%)

$\frac{\partial F}{\partial t}$ is the **local** or **Eulerian rate of change** in F

$\frac{dF}{dt}$ is the **total** or **substantial** or **material rate of change** in F

$-\vec{V} \cdot \nabla F$ is the spatial **advection** of F due to motion in the direction of gradient in F.

If an east-west oriented cold front is moving south and air parcels move together with the front adiabatically (no heating or cooling to the parcels), determine the sign of each of those terms when the above equation is apply to temperature.

(5%)

Because the motion is adiabatic, $\frac{dT}{dt} = 0$.

$$-\vec{v} \cdot \nabla T = -v \frac{\partial T}{\partial y} < 0 \text{ because } v < 0 \text{ and } \frac{\partial T}{\partial y} < 0 \text{ because } \frac{\partial T}{\partial t} = -v \frac{\partial T}{\partial y}.$$

4. (20%) An air parcel of 1 kg in mass in a tornado vortex is initially circulating at 50 m/s around the center of tornado in a circle of 1 km radius. Under the action of a central force (pressure gradient force in this case), it is brought into a circular trajectory of 500 m in radius. The frictional force and Coriolis force *due to earth rotation* can be neglected.

(a) (4%) What will be the new speed of this air parcel?

Since the force is central, the angular momentum is conserved:

$$V_1 r_1 = V_2 r_2 \rightarrow$$

$$V_2 = V_1 r_1 / r_2 = 50 \text{ m/s} \times 1000 \text{ m} / 500 \text{ m} = 100 \text{ m/s}.$$

(b) (4%) Is the kinetic energy of the air parcel conserved?

$$K_1 = \frac{1}{2} m V_1^2 = 0.5 \times 1 \text{ kg} \times 50^2 \text{ m}^2 / \text{s}^2 = 1250 \text{ J}$$

$$K_2 = \frac{1}{2} m V_2^2 = 0.5 \times 1 \text{ kg} \times 100^2 \text{ m}^2 / \text{s}^2 = 5000 \text{ J}$$

They are not equal – therefore **not conserved!**

(c) (4%) If not, what force, viewed in a reference frame rotating with the tornado, is causing this change in the kinetic energy? (Hint: Think of the earth rotating coordinate analogy).

This question is better asked like this: If not, what force is causing this change in the kinetic energy?

Answer: As the parcel moves from the trajectory of 100m radius to one with 50m radius, it has to overcome centrifugal force. It's the **pressure gradient force** (a central force) that pulls the parcel from the bigger circle to the smaller one, and in the process it does work. The PGF has to be equal to the centrifugal force. As the parcel moves towards the center, the Coriolis turns it to the right (assuming initially the circulation is counter-clockwise), converting the radial velocity created by PGF into a tangential velocity. The Coriolis force does not however create or destroy kinetic energy, however, because it is always perpendicular to

the velocity vector therefore the trajectory – it does not cause any displacement in the direction of the force therefore it does no work! The Coriolis force is an apparent force, no apparent (not real) force can do any work! Energy cannot be created or destroyed without work done by a real force.

However, because the original question might appear misleading, you will get free credit if you answered centrifugal force or Coriolis force.

(d) (4%) Is this force a true or apparent force?

The PGF is a real force. If you answered apparent force that is consistent with the answer for (d), you will get free credit.

(f) (4%) How much work does this force need to do to cause this much, if any, change in kinetic energy?

This work done equals to the change in kinetic energy (work-energy theorem), therefore

$$W = K_2 - K_1 = 5000J - 1250J = 3750J.$$

Since we said the work is done by PGF, let's see if we get the same answer from the definition of work.

The PGF is equal to the centrifugal force in magnitude but opposite in direction.

$$\vec{F} = \frac{V^2}{r} (-\hat{r}) = -\frac{V^2}{r} \hat{r}$$

$$\begin{aligned} W &= \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} = - \int_{r_1}^{r_2} \frac{V^2}{r} \hat{r} \cdot d\vec{r} = - \int_{r_1}^{r_2} \frac{V^2}{r} dr \\ &= - \int_{r_1}^{r_2} \frac{V_1^2 r_1^2}{r^3} dr = V_1^2 r_1^2 \left. \frac{1}{2r^2} \right|_{r_1}^{r_2} = \frac{(50 \times 100)^2}{2} \left(\frac{1}{50^2} - \frac{1}{100^2} \right) = 3750J \end{aligned}$$

The same as the answer from work-energy theorem.

In the above, we used $Vr = V_1 r_1$.

5. (50%) The vector equation of motion for an air parcel of unit volume can be written as

$$\frac{d\vec{V}}{dt} + 2\vec{\Omega} \times \vec{V} = -\frac{1}{\rho} \nabla p + \vec{g}_{net} \quad (1)$$

or

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho} \nabla p + \vec{g}_{net} - 2\vec{\Omega} \times \vec{V} \quad (2)$$

where \vec{V} is the earth-relative velocity.

- (a) (5%) What is the first principle / fundamental law that we use to obtain these equations?

$$\text{Newton's second law of motion, } m \frac{d\vec{V}}{dt} = \sum \vec{F}_{real}.$$

- (b) (5%) Give a physical definition for each of the terms in the equations.

$$\frac{d\vec{V}}{dt} \quad \text{the **acceleration relative to the earth**}$$

$$2\vec{\Omega} \times \vec{V} \quad \text{Coriolis acceleration due to earth rotation and earth-relative motion}$$

$$-\frac{1}{r} \nabla p \quad \text{Pressure gradient force}$$

$$\vec{g}_{net} \quad \text{apparent (net) gravity including centrifugal force due to earth rotation}$$

$$-2\vec{\Omega} \times \vec{V} \quad \text{Coriolis force, all for a unit mass.}$$

- (c) (5%) In the local Cartesian coordinate fixed to the earth, velocity $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$ and the earth's angular velocity $\vec{\Omega} = \Omega \cos(\mathbf{f})\hat{j} + \Omega \sin(\mathbf{f})\hat{k}$. Express the last term on the right hand side of Eq.(2) in its component form, i.e., in the form of $-2\vec{\Omega} \times \vec{V} = \hat{i}(\) + \hat{j}(\) + \hat{k}(\)$.

$$\begin{aligned} -2\vec{\Omega} \times \vec{V} &= -2 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Omega_x & \Omega_y & \Omega_z \\ u & v & w \end{vmatrix} \\ &= \hat{i}(2\Omega v \sin \mathbf{f} - 2\Omega w \cos \mathbf{f}) + \hat{j}(-2\Omega u \sin \mathbf{f}) + \hat{k}(2\Omega u \cos \mathbf{f}) \end{aligned}$$

- (d) (5%) Making use of the results of (c) and Eq.(2), write down the equations of motion for the three Cartesian components of velocity u , v and w . You can use $f = 2\Omega \sin(\mathbf{f})$ and $\tilde{f} = 2\Omega \cos(\mathbf{f})$ to simplify the notation.

$$\frac{du}{dt} = -\frac{1}{r} \frac{\partial p}{\partial x} + fv - \tilde{f}w$$

$$\frac{dv}{dt} = -\frac{1}{r} \frac{\partial p}{\partial y} - fu$$

$$\frac{dw}{dt} = -\frac{1}{r} \frac{\partial p}{\partial z} - g + \tilde{f}u$$

- (e) (5%) For large-scale upper-level atmospheric flows with small or zero curvature, the flow acceleration is typically much smaller than the other terms in the equations of motion and the flow is quasi-two-dimensional (i.e., w is negligibly small). If we also neglect friction,

What approximate/simplified equations can you obtain for this situation (write down the equations)?

$$0 = -\frac{1}{r} \frac{\partial p}{\partial x} + fv$$

$$0 = -\frac{1}{r} \frac{\partial p}{\partial y} - fu$$

Which is the wind velocity governed by such simplified equations called?

Geostrophic wind velocity.

- (f) (5%) In a weather chart plotted at a constant height (10km) level, the east-west pressure contours are straight and the pressure decreases northward by 1 mb over 100 km. Assuming parameter $f = 10^{-4} \text{ s}^{-1}$ and the air density there is 0.2 kg m^{-3} , determine the wind speed and direction from the simplified equations obtained in (e).

$$v = \frac{1}{f r} \frac{\partial p}{\partial x} = 0$$

$$u = -\frac{1}{f r} \frac{\partial p}{\partial y} = -\frac{1}{10^{-4} \text{ s}^{-1} \times 0.2 \text{ kg / m}^3} \frac{-100 \text{ Pascal}}{10^5 \text{ m}} = 50 \text{ m / s}$$

Therefore, the wind is **eastward** and the speed is **50m/s**.

- (g) (5%) If the pressure contours are not straight, but exhibit a trough pattern to the west and a ridge pattern to the east, and the radius of curvature is 500 km for both (pay attention to its sign), determine the gradient wind speed at the bottom of trough and the top of ridge. Note that you are dealing with normal/regular low and normal/regular high cases.

There is a problem with the numbers given for this question – with $R = 500\text{km}$, you get negative numbers inside the radical for the ridge case (okay for trough case) – not physical. It's a case where the centrifugal force plus the pressure gradient force is too large for it to be balanced by the Coriolis force. $R = 5000\text{km}$ is a more realistic number (this also shows why pressure contours are rarely very tight in high-pressure systems – we assume that PGF is the same for both ridge and trough region in this problem). If you showed that you knew how to solve the

equation, and plugged in the right numbers, your answer is considered correct. We will use $|R| = 5000$ km in our solution.

Use the given equations in natural coordinates to solve for the gradient wind speed.

$$\frac{V^2}{R} = -\frac{1}{r} \frac{\partial p}{\partial n} - fV \rightarrow V^2 + \frac{R}{r} \frac{\partial p}{\partial n} + fRV = 0$$

$$V = -\frac{fR}{2} \pm \left(\frac{f^2 R^2}{4} - \frac{R}{r} \frac{\partial p}{\partial n} \right)^{1/2}$$

For the trough, $R > 0$, need the positive sign otherwise $V < 0$, no allowed.

$$\begin{aligned} V &= -\frac{fR}{2} + \left(\frac{f^2 R^2}{4} - \frac{R}{r} \frac{\partial p}{\partial n} \right)^{1/2} \\ &= -0.5 \times 10^{-4} \times 5 \times 10^6 + \left(\frac{(5 \times 10^{-4} \times 10^6)^2}{4} - \frac{5 \times 10^6}{0.2} \frac{(-100)}{10^5} \right)^{1/2} \\ &= \mathbf{45.8 \text{ m/s.}} \end{aligned}$$

For the ridge, $R < 0$.

$$\begin{aligned} V &= -\frac{fR}{2} \pm \left(\frac{f^2 R^2}{4} - \frac{R}{r} \frac{\partial p}{\partial n} \right)^{1/2} \\ &= -0.5 \times 10^{-4} \times (-5 \times 10^6) \pm \left(\frac{(5 \times 10^{-4} \times 10^6)^2}{4} - \frac{(-5 \times 10^6)}{0.2} \frac{(-100)}{10^5} \right)^{1/2} \\ &= 250 \pm 193.6 \text{ m/s} \end{aligned}$$

The positive sign gives an unrealistic speed of 443.6 m/s and corresponding to the abnormal high case. The negative sign gives us a speed = **56.4 m/s**, the normal high case we want.

(h) (5%) Is the wind speed at the ridge larger or smaller than that at the trough?

Larger. This is expected, noting the qualitative discussion in the Notes on super-geostrophic flow that ridge and subgeostrophic flow at the trough.

Do you have flow divergence or convergence between the trough and ridge?

Divergence. Because the flow speed increases along the trajectory.

Assuming that the tropopause located above 10 km level acts like a rigid lid (i.e., $w = 0$ there), do you expect ascending or descending motion between the trough and ridge below the 10km level?

Because of the divergence, assuming the atmosphere is quasi-incompressible, there needs to be **ascending motion** to compensate the divergence at the upper-level. No descending motion can come from the above because of the rigid lid.

- (i) (5%) Assume the bottom of trough and the top of ridge are 1000 km apart, calculate the horizontal velocity divergence between the trough and ridge. Is your answer consistent with your expectation?

Using the natural coordinate, the horizontal divergence is

$$\nabla_h \cdot \vec{V}_h = \frac{\partial u}{\partial s} = \frac{u_{ridge} - u_{trough}}{1000km} = \frac{56.4 - 45.8}{1000000} = 1.06 \times 10^{-5} s^{-1}$$

Since the divergence is positive, it presents divergent flow, **consistent with** our previous discussion.

- (j) (5%) Assume the divergence remain constant between the 5 km and 10 km height levels, use the kinematic method to determine the vertical velocity at the 5 km level.

$$\text{Using } \nabla \cdot \vec{V} = \frac{\partial u}{\partial s} + \frac{\partial w}{\partial z} = 0 \rightarrow$$

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial s} = -1.06 \times 10^{-5} s^{-1}$$

$$\int_{10km}^{5km} \frac{\partial w}{\partial z} dz = \int_0^{1000km} \left(-\frac{\partial u}{\partial s} \right) ds = -\frac{u_{ridge} - u_{trough}}{1000km} = -\frac{56.4 - 45.8}{1000000} =$$

$$w(5km) = \frac{-1.06 \times 10^{-5} \times 10^6}{-5000} = 2.12 \times 10^{-3} m/s$$

Since w is positive, it represents upward/ascending motion, again consistent with the expectation.

Since the answers to (h), (i) and (j) depends on the results of (g), I will be looking for clues that you know the methods and equations to get to the right solutions and you are clear about the physical concepts.

Equations that might be useful to you:

2D equations of motion in natural coordinates: $\frac{dV}{dt} = -\frac{1}{r} \frac{\partial p}{\partial s}$ and $\frac{V^2}{R} = -\frac{1}{r} \frac{\partial p}{\partial n} - fV$.

3D divergence is $\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$. When $\nabla \cdot \vec{V} = 0$, $\frac{\partial w}{\partial z} = -\nabla_h \cdot \vec{V}_h$.

In natural coordinates, $\nabla \cdot \vec{V} = \frac{\partial u}{\partial s} + \frac{\partial w}{\partial z}$ assuming no velocity gradient in the trajectory-normal direction.

Angular momentum for 2D circular motion = ωr^2 .