Chapter VI: Motion in the 2-D Plane

Now that we have developed and refined our vector calculus concepts, we can move on to specific application of motion in the plane. In this regard, we will deal with:

- projectile motion (not much time on this)
- geopotential and central force motion
- angular momentum (important for hurricane, tornado and the atmospheric general circulations;

6.1. Conservative Force

Before we begin, let's re-introduce an important concept related to forces – conservation.

A General Definition of Conservative Force

Be definition, a force is said to be conservative if the work done by it is independent of the path, or equivalently, the work done by this force around a closed path is identically zero, i.e.,

if $\oint \vec{F} \cdot \vec{dl} = 0$, then $\vec{F}$ is said to be conservative.

Why?

Recall the Stokes' Theorem:

$$\iint (\nabla \times \vec{V}) \cdot \vec{n}\, ds = \int \vec{V} \cdot d\vec{r} \qquad (6.1)$$

this says that, if we replace $\vec{V}$ by $\vec{F}$, the force $\vec{F}$ is conservative if $\nabla \times \vec{F} = 0$ because then the right hand side – the integral along a closed path – is zero!

Therefore, one convenient way to test if a force is conservative is to see if its curl ($\nabla \times F$) is zero!

In addition, if a force can be written as the gradient of a potential function, then it is automatically a conservative force. Why? Because for any scalar ϕ, $\nabla \times \nabla f = 0$ (show it yourself!), thus, if $\vec{F} = \nabla f$, then $\nabla \times \vec{F} = 0$ automatically.

Example: Let's verify that gravity is a conservative force.

Recall that

$$\vec{F} = \frac{GMm}{r^2}\hat{r} \; . \qquad (6.2)$$

To show that it is conservative, we simply need to show that $\nabla\times\vec{F} = 0$. Let's rewrite $\vec{F}$ as

$$\vec{F} = \frac{GMm}{r^2}\frac{\vec{r}}{r} = \frac{GMm}{r^3}(x\hat{i} + y\hat{j} + z\hat{k}) \; .$$

Now, let GMm = Q, so that

$$\vec{F} = \frac{GMm}{r^2}\frac{\vec{r}}{r} = \frac{Q}{r^3}(x\hat{i} + y\hat{j} + z\hat{k}) \; .$$

Let's now compute $\nabla\times\vec{F}$ (do it on your own):

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{Qx}{r^3} & \frac{Qy}{r^3} & \frac{Qz}{r^3} \end{vmatrix}$$

Let's take only the $\hat{i}$ component:

$$\hat{i}\left[\frac{\partial}{\partial y}\left(\frac{Qz}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{Qy}{r^3}\right)\right]$$

Note that $r^3 = (x^2 + y^2 + z^2)^{3/2}$, doing the differentiation gives

$$\hat{i}\left[-3Q\frac{zy}{r^5} + 3Q\frac{yz}{r^5}\right] = 0 \; .$$

Similarly, you will find that all three components equal to zero → gravity is a conservative force!

6.2. Solving for Potential Function

Recall that in Chapter 4 when we dealt with 1-D problems, a conservative force F can be written in terms of the gradient of potential energy V, because by definition, the work done by a conservative force is a function of the spatial coordinate only. We call the work done by the force to bring an object to a standard location the potential energy:

$$V(x) = \int_{x}^{x_s} F dx = -\int_{x_s}^{x} F dx \qquad (6.2a)$$

therefore $$F = -\frac{dV}{dx}. \qquad (6.2b)$$

Similar is true for multi-dimensional problem. Analogously, we define potential energy

$$V(\vec{r}) = V(x, y, z) = -\int_{\vec{r}_s}^{\vec{r}} \vec{F} \cdot d\vec{r} \qquad (6.2c)$$

and the right hand side integration should be <u>independent of the path</u> from $\vec{r}_s \ to \vec{r}$.

Eq.(6.2c) actually came from the integration of

$$dV(\vec{r}) = -\vec{F} \cdot d\vec{r}, \qquad (6.2d)$$

which means that the change in the potential energy = the negative of work done moving the object from $\vec{r} \ to \ \vec{r} + d\vec{r}$.

Using the definition of dV and Eq.(6.2d), we have

$$dV(\vec{r}) = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy = -\left(F_x dx + F_y dy\right) \qquad (6.2e)$$

therefore

$$F_x = -\frac{\partial V}{\partial x}, \ F_y = -\frac{\partial V}{\partial y} \qquad (6.2f)$$

or $$\vec{F} = -\nabla V \qquad (6.2g)$$

Therefore, we have showed that <u>a conservative force can always be written in terms of the gradient of a scalar function,</u> and we call it the potential energy.

Given $\vec{F}$, how do we obtain the potential V? For 1-D problems, we can obtain V by simply integrate F according (6.2a). For multi-dimensional problems, the method is analogous though a little more complicated. We show this using an example.

Example: Let $\vec{F} = Ax^2\hat{i} + By\hat{j} = F_x\hat{i} + F_y\hat{j}$. (6.3)

If you take $\nabla\times\vec{F}$, you will find that it is zero, and thus the force is conservative and can be written as

$$\vec{F} = -\nabla V \tag{6.4}$$

then

$$F_x = -\frac{\partial V}{\partial x},\ F_y = -\frac{\partial V}{\partial y} \tag{6.5}$$

therefore

$$-\frac{\partial V}{\partial x} = Ax^2\ . \tag{6.6}$$

Integrate it with respect to x, we get

$$V = -Ax^3/3 + f(y) \tag{6.7}$$

where $f(y)$ is the integration "constant" which can still be a function of y. The partial derivative of $f(y)$ with respect to x goes to zero.

How do you determine $f(y)$? Notice we haven't used the second equation,

$$F_y = -\frac{\partial V}{\partial y} = By \tag{6.8}$$

yet. The solution V should also satisfy this equation. Substituting (6.7) into (6.8):

$$-\frac{\partial[-Ax^3/3 + f(y)]}{\partial y} = By$$

→
$$-\frac{\partial f(y)}{\partial y} = By \tag{6.9}$$

→
$$f(y) = -\frac{By^2}{2} + C \tag{6.10}$$

Here, C is truly a constant because f = f(y) only, so it can't a function of either x or y. And this constant can be arbitrarily chosen since for the force, it is the gradient of the potential function that matters. C is usually chosen based on convention. For example, for the gravity, we often choose V = 0 at the sea level, which determines C.

The final solution for V is

$$V = -\frac{Ax^3}{3} - \frac{By^2}{2} + C \tag{6.11}$$

6.3. Projectile Motion

Projectile motion itself has little to do with the atmosphere – we examine it nonetheless because it provides an excellent, simple example of solving coupled equations of motion. Until now we have only looked at 1-D motion. Of course, meteorologists are involved in ballistic research when it comes to clouds, frictional properties of air and aerosols, etc.

The vector equation of motion governing projectile is

$$\frac{d(\vec{V}m)}{dt} = \sum \vec{F} \quad . \tag{6.12}$$

For a single projectile with mass m, subjecting to gravity only, the equation is

$$m\frac{d^2\vec{r}}{dt^2} = -mg\hat{k} \ . \tag{6.13}$$

In reality, this includes 3 equations that can be written in component form as follows:

$$\begin{aligned} m\frac{d^2x}{dt^2} &= 0, \\ m\frac{d^2y}{dt^2} &= 0, \\ m\frac{d^2z}{dt^2} &= -mg \end{aligned} \tag{6.14}$$

In this simplest case with gravity only (no friction etc.), each of the three equations can be solved independently. The solutions are:

$$\begin{aligned} x &= x_0 + u_0 t \\ y &= y_0 + v_0 t \\ z &= z_0 + w_0 t - gt^2/2 \end{aligned} \tag{6.15}$$

Therefore, the objection moves in the horizontal directions at constant velocities u_0 and v_0, and in the vertical, it behaves exactly like a 1-D upward thrown object.

Let's assume that the projectile starts at (x, y, z) = (0, 0, 0) with motion only in the x-z plane → $v_0 = 0$. Then,

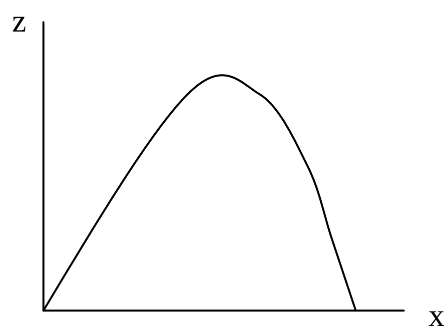
$$x = u_0 t$$
$$y = 0 \quad (6.16)$$
$$z = w_0 - gt^2/2$$

That's it! If we solve the first equation above for t and use it in the 3rd equation, we have:

$$z = \frac{w_0}{u_0}x - \frac{1}{2}\frac{gx^2}{u_0^2} \quad (6.17)$$

which tells you the height of the projectile at any x location instead of at any time.

Equation (6.17) is a parabolic equation, which describes the projectile trajectory, as shown blow.



We can find that maximum height of the trajectory by computing dz/dx and setting it to zero. Doing this:

$$\frac{dz}{dx} = \frac{w_0}{u_0} - \frac{gx}{u_0^2} = 0 \rightarrow$$

$$x_{\max} = \frac{w_0 u_0}{g}$$ - the x position of the peak of the trajectory.

The peak is reached at $$t_{\max} = \frac{x}{u_0} = \frac{w_0}{g},$$

and the peak height is $$z_{\max} = \frac{w_0}{u_0}\frac{w_0 u_0}{g} - \frac{1}{2}\frac{g}{u_0^2}\left(\frac{w_0 u_0}{g}\right)^2 = \frac{w_0^2}{2g}.$$

The peak height is actually independent of the horizontal component of the initial velocity! The same peak height will be reached if the object is projected vertically at initial speed of w_0 (with zero initial horizontal velocity)!

Often the projectile problem will be given in terms of the angle at which the projectile is fired. You need to compute u_0 and w_0 from the initial speed and angle.

What has this analysis neglected?

- air resistance (friction)
- air motion
- the earth's rotation – a big thing for long-range ballistic missiles

These additional effects make the three components equations coupled to each other, but we will not go into details here. The last topic will be covered in next chapter.

In the following, we will look at another special type of motion before getting into angular momentum, the motion under the influence of central force.

6.4. Central Force Motion

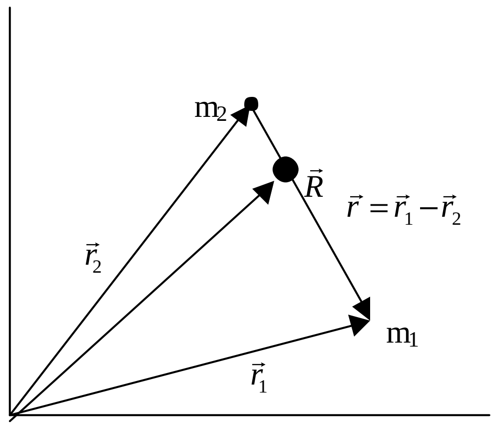
The motion of a central force, which we will define momentarily, arose out of Newton's law of gravitation and his successful use in explaining Kepler's elliptic orbits for the planets.

A central force is one in which the force has the form:

$$\vec{F} = F(r)\hat{r}, \tag{6.18}$$

i.e., the force acts along a line connecting one or more bodies and is somehow related to the distance between them. Examples include gravitational attraction, the force that causes an electron to orbit the nucleus of an atom, etc. In most physically important cases, the force follows an "inverse square law", i.e., $F \sim 1/r^2$. Two interacting bodies is a simple matter, but more than two is extremely complex … referenced to as the "n-body problem" where n > 2.

To keep life simple, consider 2 bodies, in isolation, that are under the influence of a central force. Let the masses be m_1 and m_2 and their position vectors be $\vec{r}_1$ and $\vec{r}_2$:



Let $\vec{r} = \vec{r}_1 - \vec{r}_2$ and r = | $\vec{r}$ |. The equations of motion for the two bodies are

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = f(r)\hat{r}$$
$$m_2 \frac{d^2\vec{r}_2}{dt^2} = -f(r)\hat{r} \qquad (6.19)$$

The force is attractive for f(r) < 0 and repulsive for f(r)>0. In reality, both objects are attracted to one another, but this is a mathematically convenient way to deal with the force… think of Newton's law of gravitation for two bodies.

Note that these two equations of motion are coupled via $\vec{r}$. It turns out that life is simpler if we replace $\vec{r}_1$ and $\vec{r}_2$ by $\vec{r}$ and by the vector to the **center of mass** of the 2 bodies:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \qquad (6.19a)$$

(Note, for an N body system, the center of mass is defined as $\vec{R} = \frac{\sum_i \vec{r}_i m_i}{\sum_i m_i}$)

The center of mass is special – the system behaves as if all the mass is concentrated at this single point. For the above 2-body system, since there is no external force (all forces are internal interacting forces), therefore the equation of motion for $(m_1 + m_2)$ at $\vec{R}$ is:

$$(m_1+m_2)\frac{d^2\vec{R}}{dt^2}=0 \rightarrow \tag{6.19b}$$

$$\vec{R}=\vec{R}_0+\vec{V}t\,,$$

which is the position vector of the center of mass.

What about the positions of the individual bodies? To find them, we will see that it's easier to first fine $\vec{r}$. Recall our original equations of motion in (6.19) for $\vec{r}_1$ and $\vec{r}_2$. If we divide them by m_1 and m_2, respectively, and subtract the two resultant equations, we have

$$\frac{d^2\vec{r}_1}{dt^2}-\frac{d^2\vec{r}_2}{dt^2}=\left(\frac{1}{m_1}+\frac{1}{m_2}\right)f(r)\hat{r}$$
$$\Rightarrow \frac{m_1m_2}{m_1+m_2}\left(\frac{d^2\vec{r}_1}{dt^2}-\frac{d^2\vec{r}_2}{dt^2}\right)=f(r)\hat{r}$$

If $m\equiv\dfrac{m_1m_2}{m_1+m_2}$ = reduced mass (definition), and with $\dfrac{d^2\vec{r}_1}{dt^2}-\dfrac{d^2\vec{r}_2}{dt^2}=\dfrac{d^2\vec{r}}{dt^2}$, we have

$$m\frac{d^2\vec{r}}{dt^2}=f(r)\hat{r}\,. \tag{6.20}$$

Equations (6.19b) and (6.20) <u>fully</u> describe the system. Unlike the coupled equations in (6.19), these two are uncoupled and are analogous to the equation for a 1-particle system! Note that one cannot do this for 3 or more particles, and to this day no solution exists for those system!

Goal: We now want to solve this differential equation for $\vec{r}(t)$, and from that we can easily get $\vec{r}_1$ and $\vec{r}_2$:

$$\vec{r}_1=\vec{R}+\frac{m_2}{m_1+m_2}\vec{r},$$
$$\vec{r}_2=\vec{R}-\frac{m_1}{m_1+m_2}\vec{r}$$

where is the final solution to the two body problem!

<u>**SHOW THIS FOR YOURSELVES!**</u>

6.5. Angular Momentum and Its Conservation

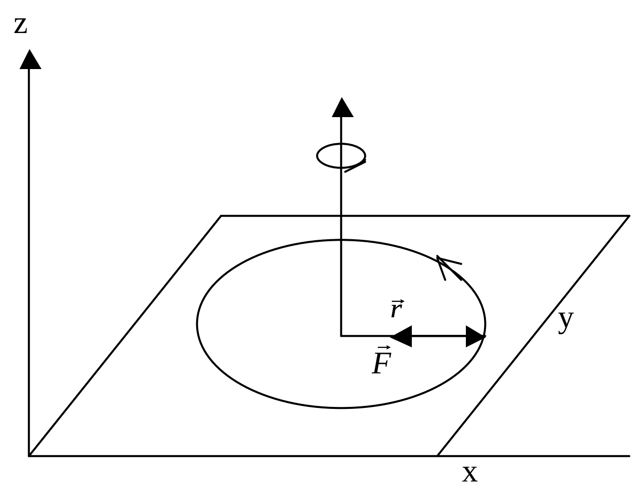
We can now use the general equation

$$m\frac{d^2\vec{r}}{dt^2} = f(r)\hat{r} \qquad (6.21)$$

to look at all sorts of motion involving one or many masses. (If we have one mass, then μ = m). As an example, consider motion in a plane as preface to the topic of angular momentum.

Definition: Torque (a vector) = $\vec{r}\times\vec{F}$. Torque causes momentum to change, while force causes regular momentum to change. In a 2-D plane, the torque (also called moment of force) about an origin = radius $r \times$ the component of the force perpendicular to the radius vector, consistent with the above vector definition.

For simplicity, consider a single particle of mass m subjecting to a central force:



Because this is central force motion, $\vec{F}$ lies in the plane along $\vec{r}$ - and thus $\vec{F}$ can excert no torgue (= $\vec{r}\times\vec{F}$) on mass m ($\vec{F}$ and $\vec{r}$ are parallel).

Angular Momentum

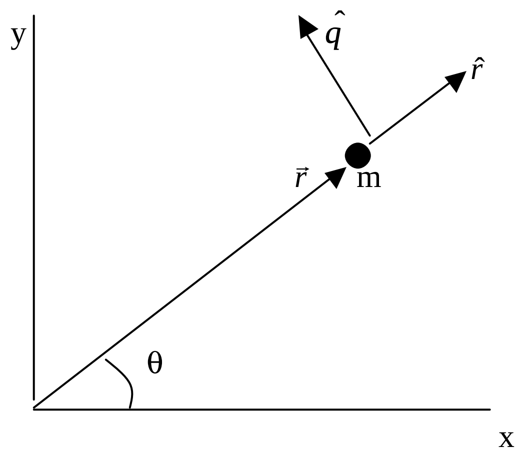
For mass moving about a point (origin) O, the **vector angular momentum** is defined as

$$\vec{L} = \vec{r} \times m\vec{V}\ . \tag{6.22}$$

where $\vec{r}$ is the position vector from O to the particle with mass *m* and velocity $\vec{V}$.

Velocity and Acceleration in Plane Polar Coordinates

For a rotational motion in a 2-D plane, it's easier to work in plane polar coordinates.



In plane polar coordinates, the unit vectors are $\hat{r}$ and $\hat{q}$. Let's now express our equation of motion in these coordinates:

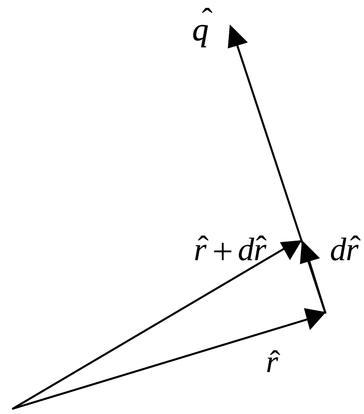
$$m\frac{d^2\vec{r}}{dt^2} = f(r)\hat{r} \tag{6.23}$$

From the diagram,

$$\vec{r} = r\,\hat{r}(q)$$

where the unit vector $\hat{r}$ is a function of θ.

$$\therefore \qquad \frac{d\vec{r}}{dt} = \hat{r}\frac{dr}{dt} + r\frac{d\hat{r}}{dt} = \vec{V} \tag{6.23a}$$



The change in $\hat{r}$ as a result of change in θ by the amount of dθ in the in direction of $\hat{q}$, according to the above figure, and is given by

$$d\hat{r} = |\hat{r}|\, dq\, \hat{q} = dq\, \hat{q}$$

therefore

$$\frac{d\hat{r}}{dq} = \hat{q}$$

thus (6.23a) becomes

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + \left(r\frac{dq}{dt}\right)\frac{d\hat{r}}{dq} = V_r\hat{r} + V_q\hat{q}$$

therefore

$$\boxed{\vec{V} = V_r\hat{r} + V_q\hat{q}} \qquad (6.24)$$

where V_r the <u>radial component</u>, and V_θ is the <u>tangential or azimuthal</u> component.

One can differentiate the velocity as well to obtain the acceleration, which, of course, is important if we want to apply $\vec{F} = m\vec{a}$.

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{dq}{dt}\right)^2\right]\hat{r} + \left[r\frac{d^2q}{dt^2} + 2\frac{dr}{dt}\frac{dq}{dt}\right]\hat{q}$$

Now let's use this in our central force equation of motion:

$$m\frac{d^2\vec{r}}{dt^2} = f(r)\hat{r}\ .$$

Equating "like" components on each side of this vector equation, and using the convention that a 'dot' denotes d( )/dt, we have

$$m(\ddot{r} - r\dot{\theta}^2) = f(r) \qquad (6.25a)$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \qquad (6.25b)$$

They are basically equations of motion (based on Newton's second law) applied in the radial and azimuthal directions, respectively.

They are coupled second-order differential equations in time, are not very easy to solve.

Angular Momentum Conservation

Recall that the direction of the angular momentum is a constant, i.e., the motion itself is in the x-y plane. There are 2 other important quantities in this system:

- the magnitude of the angular momentum;
- the total energy.

Let's look only at the former – recall that conservation properties are very important in meteorology, and that conservation means that d/dt = 0 (Lagrangian derivative following the motion).

Let's first compute the magnitude of the angular momentum:

$$\begin{aligned}|\vec{L}| &= m|\hat{r}\times\vec{V}|\\ &= m|\hat{r}||\vec{V}|\sin(90^\circ)\\ &= mrV_\theta\\ &= mr(r\dot{\theta})\end{aligned}$$

$$\therefore \quad \boxed{|\vec{L}| = mr^2\dot{\theta}} \qquad (6.26)$$

Is the property conserved following the motion?

What we are about to do is something you will see a lot in meteorology – taking the basic equations of motion (here, in plane polar coordinates with a central force) and manipulating then to obtain a conservation equation.

If we want to show that $|\vec{L}| = mr^2\dot{q}$ is conserved, we want to show that

$$\frac{d}{dt}(mr^2\dot{q}) = 0.$$

Expanding gives

$$m(2r\dot{r}\dot{q} + r^2\ddot{q}) = 0. \qquad (6.27)$$

Going back to the equations of motion, multiplying Eq.(6.25b) gives

$$m(r^2\ddot{q} + 2r\dot{r}\dot{q}) = 0$$

which is exactly the same (6.27). Therefore $\frac{d}{dt}|\vec{L}| = 0$.

So, we have shown that **ANGULAR MOMENTUM IS CONSERVED FOLLOWING THE MOTION** .

But, when is this true? Only when the torque is zero, or in other words, when the applied force acts along (parallel to) the position vector of the particle (see previously drawing). This is true when every force acting on the particle is a central force.

This property is very important in rotating atmosphere flows and is the "ice skater" analogy that you have heard so much about.

Let's look closer:

If angular momentum is conserved, then

$$\frac{d}{dt}(mr^2\dot{q}) = 0$$

or

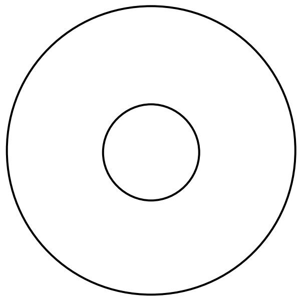
$$\frac{d}{dt}(mrV_q) = 0$$

→

$$(mrV_q)_{\text{point1}} = (mrV_q)_{\text{point2}},$$

where, via interpretation of the Lagrangian derivative, points 1 and 2 are separated in space by some interval of time.

Example: Suppose an air parcel is rotating around a mesocyclone of radius 2 km with a tangential speed of 30 m/s. How fast will the parcel be going if it spirals into a radius of 0.5 km?



$$(r V_\theta)_1 = (r V_\theta)_2$$

$$(2000\text{m}) (30 \text{ m/s}) = (500 \text{ m}) (V_\theta)_2$$

$$(V_\theta)_2 = 120 \text{ m/s!}$$

The point is that, as a parcel decreases its distance to the axis of rotation, its tangential speed must increases proportionally to keep A.M. constant.

Angular Momentum Theorem

Let's now consider pure rotation about the vertical axis --- recall vertical vorticity and its importance in meteorology. With the vertical axis of rotation fixed, we can look at the vertical component of A.M.

$$L_z = mr^2\dot{q} = Iw$$

where $I \equiv mr^2$ = moment of inertia . It depends on the distribution of mass and the distance of the mass from the axis of rotation. $w \equiv \dot{q}$.

Let's now look at the general 3-D case:

Now, $\vec{t} = \vec{r} \times \vec{F} = \text{torgue}$

and $\vec{L} = \vec{r} \times m\vec{V}.$

$$\frac{d\vec{L}}{dt} = m\frac{d}{dt}(\vec{r}\times\vec{V})$$
$$= m\left[\frac{d\vec{r}}{dt}\times\vec{V}\right] + m\left[\vec{r}\times\frac{d\vec{V}}{dt}\right]$$
$$= m\left[\vec{V}\times\vec{V}\right] + \left[\vec{r}\times m\vec{a}\right]$$
$$= \vec{r}\times\vec{F} = \vec{t}$$

Thus, $$\boxed{\frac{d\vec{L}}{dt} = \vec{t}}$$

Therefore the <u>time rate of change in angular momentum equals the torque</u> → torque causes angular momentum to change, analogous to the fact that force causes momentum to change. The above is the <u>Angular Momentum Theorem!</u>

As we saw earlier, the A.M. is conserved ($\frac{d\vec{L}}{dt} = 0$) if the torque is zero. The above equation quantifies this fact! Again, note the Lagrangian derivative.

Let's return briefly to the z component of A.M.

$L_z = I\,w.$

By the above, we know that

$$t_z = \frac{dL_z}{dt} = \frac{d}{dt}(Iw) = \overset{0}{\cancel{\frac{dI}{dt}}}w + I\frac{dw}{dt} = Ia$$

→ $$\boxed{t_z = Ia}$$

where α = angular acceleration = $d^2\theta/dt^2$. This equation looks very much like F = m a, though is applicable to bodies undergoing rotary motion. We can define the kinetic energy here as 1/2 $I\omega^2$ (I is analogous to mass).

We have the following correspondences:

τ → F
I → m
α → a