# Chapter V: Review and Application of Vectors

In the previously chapters, we established the basic framework of mechanics, now we move to much more realistic problems in multiple dimensions. This will allow us to examine rotational motion, plane motion, and much more realistic forces. First, we will need to review the basics of vector calculus.

## *5.1. Vector Algegra*

(read p72-90 in Symon)

A vector is a directed line segment that has both **magnitude** and **direction** - Both are necessary to specify a vector. We indicate a vector as $\vec{A}$. Some times ***A*** is used instead.

**Basic Properties and Definitions:**

**1).** If 2 vectors have the same length and direction, they are said to be equal:

$$\vec{A} = \vec{B} \ or \ \vec{B} = \vec{A} \tag{5.1}$$

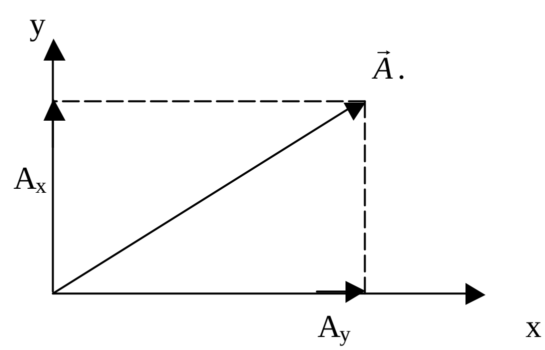
**2).** We can use vectors independent of their coordinate system, e.g.,

$$\vec{z} \equiv \nabla \times \vec{V} \tag{5.2}$$

refers to the **same** vector no matter what coordinate system (e.g., Cartesian or polar coordinate) you use.

**Vector Components in a Given Coordinate:**

But at some point, we will want to look at specific results, and this requires that we specify a coordinate system and the components of a vector. These are basically projections of a vector along the coordinate axes. Consider a 2-D example:



Thus, we see that $A_x$ and $A_y$ are the projections if $\vec{A}$ along the x and y coordinate axes, respectively.

**Unit or Base Vectors and Magnitude/Length of Vector:**

To write $\vec{A}$ in terms of these two vectors, we need to define the unit vectors. Unit vectors are also called base vectors.

Before going further, we need to first define the magnitude of a vector, $|\vec{A}|$. This is basically the **length** of vector. A base vector or unit vector is thus

$$\frac{\vec{A}}{|\vec{A}|} = \hat{A} .$$

It points in the direction of $\vec{A}$ with amplitude = unity.

By convention, the unit vector in a 3-D Cartesian framework are $\hat{i}, \hat{j}, \hat{k}$ in the x, y and x directions, respectively. With this concept, we can now write

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \qquad (5.3)$$

(note that a vector can only be equal to a vector, not scalar).

$A_x$, $A_y$ and $A_z$ are called the components of the vector $\vec{A}$. Often you will see this written more compactly as ($A_x$, $A_y$, $A_z$). Note that, if $\vec{A} = \vec{B}$, then $A_x = B_x$, $A_y = B_y$, and $A_z = B_z$.

**3)** We define the magnitude of a vector as

$$|\vec{A}| \equiv \sqrt{A_x^2 + A_y^2 + A_z^2} . \qquad (5.4)$$

This is also called the modulus.

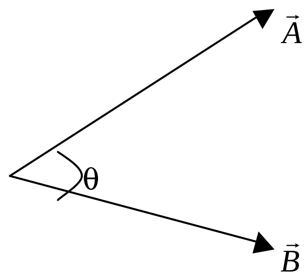
The ability to manipulate vectors is critical for meteorology. On p73-76 of Symon book (see handout), the basic algebra of vectors is discussed – read this very carefully! Make sure you can add + subtract vectors. We will spend time in class going over the more complicated aspects of vector manipulations.

**4).** Scalar, Dot or Inner Product

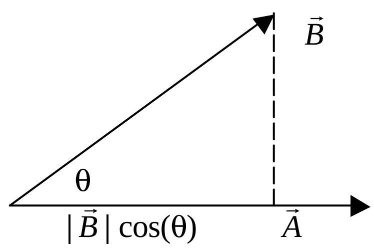
If $\vec{A}$ and $\vec{B}$ are 2 arbitrary vectors (could be in any coordinate), then the inner product is defined as

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos(\theta) \qquad (5.5)$$

where θ is the angle between $\vec{A}$ and $\vec{B}$:

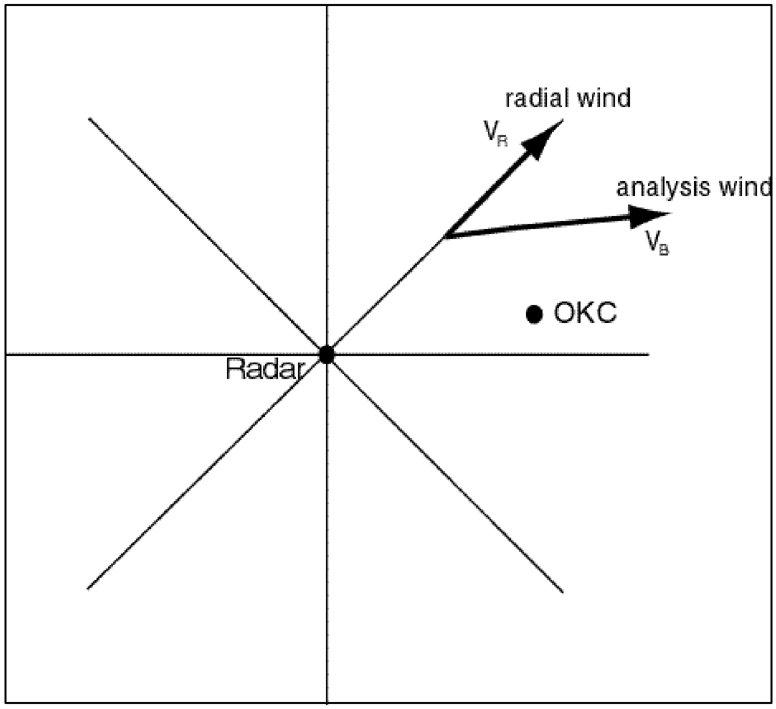



Note that $\vec{A} \cdot \vec{B}$ = a <u>scalar</u>. Physically, one can view the dot product as the <u>projection</u> of one vector onto another.




The dot product $\vec{A}$ and $\vec{B}$ = the magnitude of $\vec{A}$ times the projection of $\vec{B}$ onto $\vec{A}$.

When is this useful? Consider a 2-D wind analysis. Suppose we have a Doppler radar in the region – giving very high-resolution wind measurements, but only of wind component parallel to the radar beam.




An initial analysis of wind ($\vec{V}_B$) is first performed using coarse resolution convection observations, without radar data. To make use of the radar data, we first project $\vec{V}_B$ to the radial direction and compare this component with radial velocity $\vec{V}_R$ (the only velocity component that the Doppler radar can see), if they are equal, then the initial analysis is considered perfect. If they don't, certain adjustment is made to $\vec{V}_B$ so that the projection matches $\vec{V}_R$.

If you plan to do any sort of work with radars, you need to have a solid understanding of vectors and of spherical geometry! Of course, even you don't work with radar, you still need to know vectors very well to study meteorology.

Note that, if $\vec{A}\cdot\vec{B}=0$, then $\vec{A}\perp\vec{B}$. A special case of this is that one or both of the vectors is/are zero.

Also, $\vec{A}\cdot\vec{A}=|\vec{A}|^2=\sqrt{\vec{A}\cdot\vec{A}}$. Verify that this fits our earlier definition of the magnitude of a vector.

In terms of components,

$$\vec{A}\cdot\vec{B}=A_xB_x+A_yB_y+A_zB_z. \tag{5.6}$$

Note also that

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot k = 1$$
$$\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0$$
$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

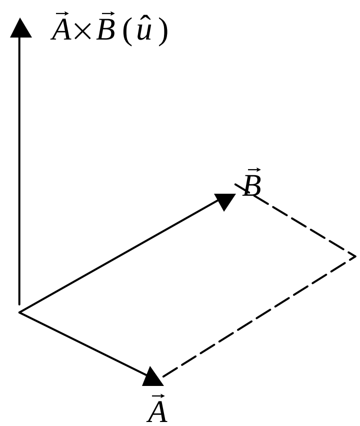
**5)** Vector or "cross" or outer product

The outer product between 2 arbitrary vectors $\vec{A}$ and $\vec{B}$ is defined as

$$\vec{A}\times\vec{B} = AB\sin(q)\,\hat{u} = \vec{C} \;\; = \text{a vector} \qquad (5.7)$$

where $\hat{u}$ is the unit vector indicating the direction of $\vec{A}\times\vec{B}$. In contrast to the inner product, which yields a scalar, the cross or outer product yields a vector!

It too has a simple geometric definition:




The direction of $\vec{A}\times\vec{B}$ is given by the right hand rule – it is $\perp$ to the plane containing $\vec{A}$ and $\vec{B}$. Note also that $|\vec{A}\times\vec{B}|$ = area of the parallelogram containing $\vec{A}$ and $\vec{B}$.

If $\vec{A}=\vec{B}$ or $\vec{A}\,||\,\vec{B}$, then $\vec{A}\times\vec{B}$ =0. This is a useful way to see if 2 vectors are parallel.

See p79-80 of Symon (handout) for useful identities with the cross product. The most common is

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}.$$

## Physical Examples of Dot and Cross Products

The most common example of the dot product is in the definition of work:

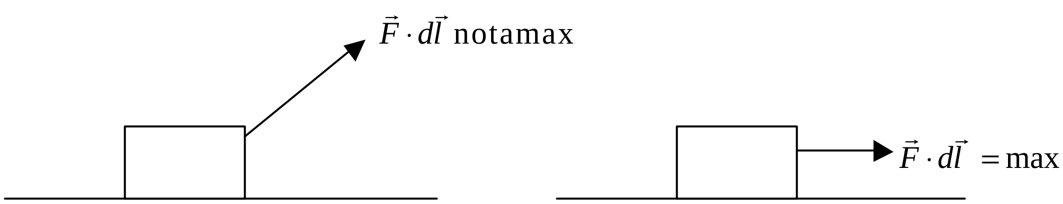
$$W \equiv \int \vec{F} \cdot \vec{dl}$$

(in 1-D, this was $\int Fdx$ )

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$
$$d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

This says that the work is the integral of the projection of the force in the direction of the displacement. Force applied in the direction without motion does not lead to work! Only the force applied in the direction of motion does work.

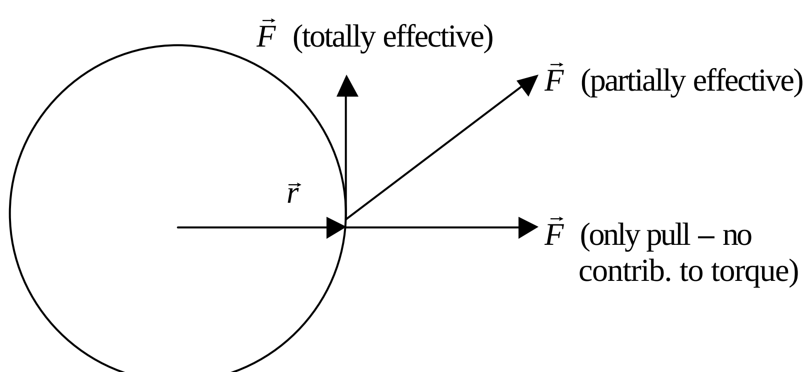
Think of dragging a heavy box along the floor with a rope. Which is the most effective strategy that maximizes work?




Examples of the cross product also abound, and the most common is torque $\vec{t}$ :

$$\vec{t} \equiv \vec{r} \times \vec{F}$$

where $\vec{r}$ is the vector from the axis of rotation to the point at which the force is applied.



Note the direction of $\vec{r} \times \vec{F}$ -- it is upward along the axis of the cylinder if we are looking down from above.

How do we find the cross product? The easiest way is via the determinant rule. If we are in a Cartesian coordinate system with

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$, we can write

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \qquad (5.8)$$

All we are doing is expand about the top row

$$\vec{A}\times\vec{B} = \hat{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= \hat{i}\,(A_yB_z - A_zB_y) - \hat{j}(A_xB_z - A_zB_x) + \hat{k}(A_xB_y - A_yB_x)\ . \qquad (5.9)$$

Another useful thing to remember is the scalar triple product

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \qquad (5.10)$$

And you proceed to expand as before.

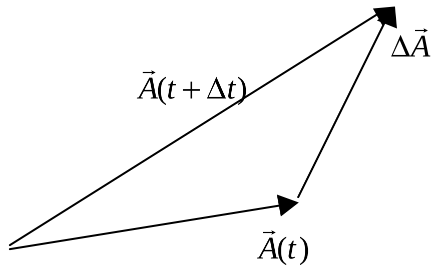
## *5.2. Differentiation and Integration of Vectors*

**Differentiation of Vectors**

It's clear that we will need to differentiate and integrate vectors as well – this is very important in fluid mechanics (e.g, the acceleration vector = total time derivative of the velocity vector). Let's focus first on differentiation.

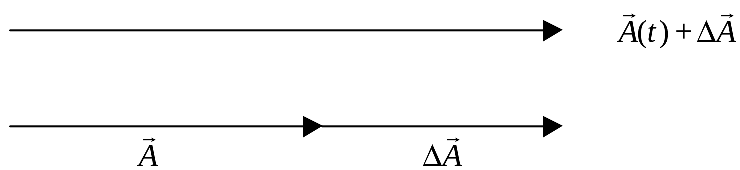
Let's assume that $\vec{A} = \vec{A}(t)$ only. Then, recall our definition of a derivative:

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t) + \vec{A}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\vec{A}(t+\Delta t)}{\Delta t}$$



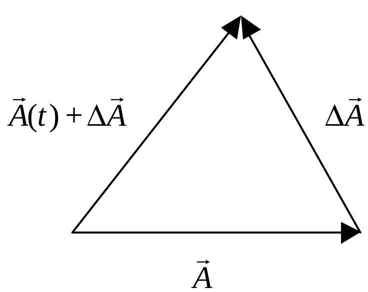

Verify that $\vec{A}(t) + \Delta\vec{A}(t) = \vec{A}(t+\Delta t)$ yourself! Note that $d\vec{A}/dt$ is also a <u>vector!</u> The new twist here is that $\vec{A}$ can have a <u>derivative</u> or <u>change</u> due to <u>2 things</u> : a change in <u>direction</u> or <u>change</u> due to <u>magnitude</u>. Let's look more closely:

**CASE I:**




direction remains the same, only the magnitude has been altered

**CASE II**:




only the direction has changed – the magnitude stayed the same.

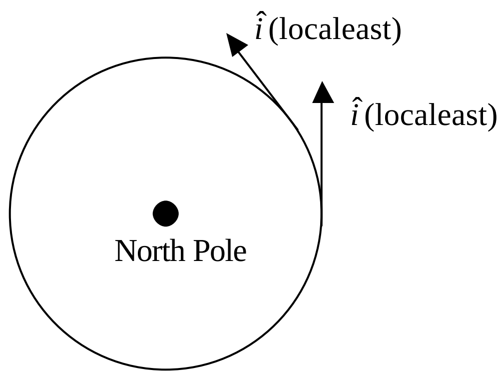
Additional change in a vector can be due the change/motion of the coordinate itself. In that case, the unit/base vectors are not longer constant, leading to additional terms. The total time derivative is then

$$\frac{d\vec{A}}{dt} = \frac{d}{dt}[\hat{i}A_x + \hat{j}A_y + \hat{k}A_z]$$

$$= \hat{i}\frac{dA_x}{dt} + \hat{j}\frac{dA_y}{dt} + \hat{k}\frac{dA_z}{dt} \quad \}\text{change of mag part} \qquad (5.11)$$

$$+ A_x\frac{d\hat{i}}{dt} + A_y\frac{d\hat{j}}{dt} + A_z\frac{d\hat{k}}{dt} \quad \}\text{change of dir part}$$

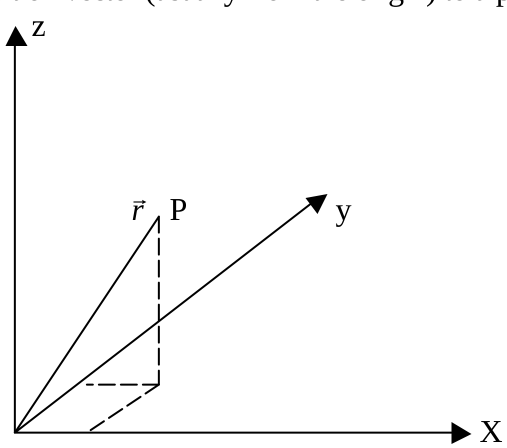
This is extremely important in meteorology because our coordinate system $(\hat{i}, \hat{j}, \hat{k})$ is rotating with time. Consider looking down on the earth from above:




The definition of "east" depends on where you are at and the fact that the earth is rotating. Here, $\hat{i}$ is not constant in terms of its direction, and thus it's easy to see that the earth is a non-inertial (accelerating) reference frame. It is through $\frac{d\hat{i}}{dt}$ etc that the Coriolis force arises, and we will look at this before too long.

At this point, we can bring in the notation of a position vector to show how we get to velocity and acceleration.

The position vector allows us to locate a point in space once we define our coordinate system. Let's let $\vec{r}$ = position vector (usually from the origin) to a point P:



Here, $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ . One can define $\vec{r}$ w/r/t any coordinate system – you just have to specify it.

If a particle moves from point P to another point Q in some interval Δt, then we can compute its velocity as

$$\vec{V} = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$$

where $\vec{r}$ (t) = position vector at location P

$\vec{r}$ (t+Δt) = position vector at location Q.

In meteorology, the convention is that

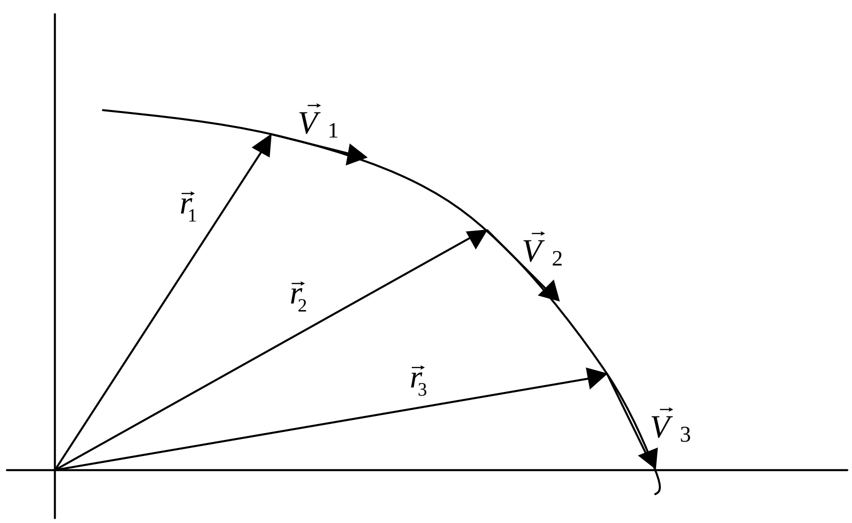
$$\vec{V} = \frac{d\vec{r}}{dt} = \hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt} = u\hat{i} + v\hat{j} + w\hat{k} \qquad (5.12)$$

$u \equiv \frac{dx}{dt} =$ east-west velocity (+east)

$v \equiv \frac{dy}{dt} =$ north-south velocity (+north)

$w \equiv \frac{dz}{dt} =$ vertical velocity (+up)

Note that $\vec{V}$ is everywhere tangent to the trajectory of the particle:



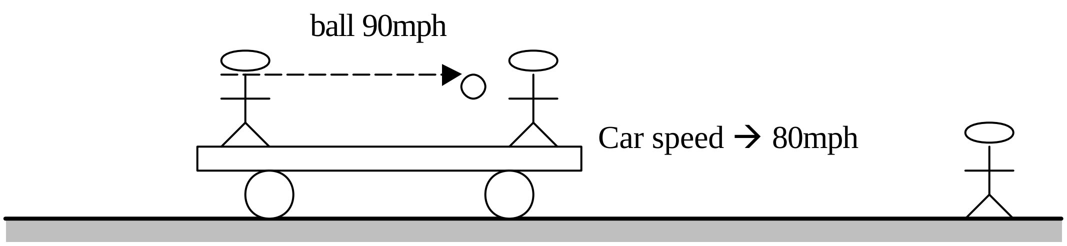
Similarly, the acceleration is given by

$$\frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2} = \hat{i}\frac{du}{dt} + \hat{j}\frac{dv}{dt} + \hat{k}\frac{dw}{dt}. \qquad (5.13)$$

**Absolute and Relative Motion**

Suppose we are in a coordinate system that is moving. Are the $\vec{r}, \vec{V}, \text{and}\, \vec{a}$ the same?

Consider the following example:



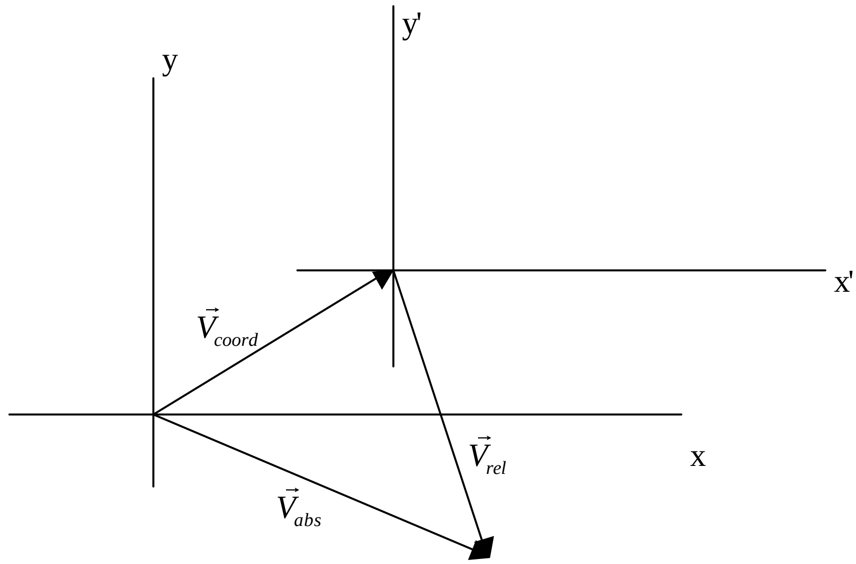

Speed of rail car = 80 mph
Speed of baseball = 90 mph relative to observer on the car
Speed of baseball relative to person on the ground? 80+90!

Motion is always measured with respect to some reference point, e.g., the ground, 'fixed' stars, etc. in the above example, the "catcher" on the train and the catcher on the ground obviously fell a different impact! Physically we see that

$V_{ball}$(rel to ground) = $V_{ball}$(rel to train) + $V_{ball}$(train)

If we take the gound as being the "absolute" or reference point, then

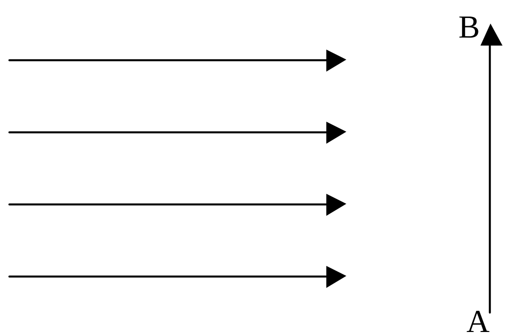
$$\vec{V}_{abs} = \vec{V}_{rel} + \vec{V}_{coord}$$




Let (x,y) be the "fixed" or absolute coordinate system
Let (x', y') be the moving coordinate system

Is this important in meteorology? Absolutely! The flow relative to the storm is the only thing that matters for storm development – usually the S-R flow in the lower 3km. It is the storm-relative low-level flow that feeds the updraft with low-level moisture!

Other examples – airplane flowing in an air stream




If an airplane wants to fly from A to B, which way will it have to head in order to make it?

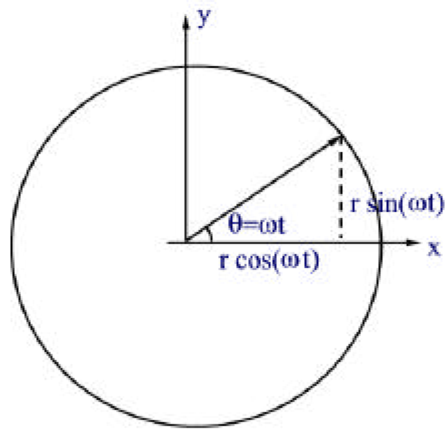
$V_{abs}$ = ground relative velocity
$V_{coord}$ = wind speed (because the airplane is in the moving air)
$V_{rel}$ = plane flying through the air (moving coordinate) needs to know how it should fly relative to the wind.

$\vec{V}_{abs} = \vec{V}_{coord} + \vec{V}_{rel}$ is the ground relative speed

## **Uniform Circular Motion**

One simple example that is useful for illustrating vector motion is uniform circular motion (UCM). Clearly rotation is important in the atmosphere, so this is a good place to start!

Consider a particle moving in a circle in the x-y plane – like a tornado or a hurricane (looking down from above):




If ω is the angular rotation rate in radians/s, then θ = ωt is the angle moved through in time t.

From the diagram, we see that

$$\vec{r} = \hat{i}\, r\cos(wt) + \hat{j} r\sin(wt)$$

thus, $|\vec{r}| = [r^2\cos^2(wt) + r^2\sin^2(wt)]^{1/2} = r = \text{consant}$

Clearly the trajectory is a circle – the length of $\vec{r}$ does not change with time, but the position or direction does → the particle is accelerating because it is moving in a circle. So if we compute $d^2\vec{r}/dt^2$, it had better not be zero!

What is the velocity?

$$\vec{r} = \hat{i}\, r\cos(wt) + \hat{j} r\sin(wt)$$

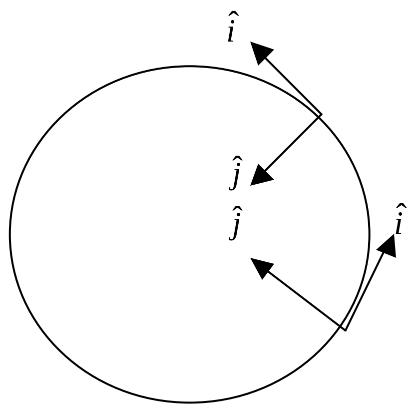
$$\vec{V} = \frac{d\vec{r}}{dt} = rw[-\hat{i}\sin(wt) + \hat{j}\cos(wt)]$$

$$(\text{note } |\vec{V}| = rw = \text{constant})$$

If you take $\vec{V} \cdot \vec{r}$ , you will see it equals zero and thus $\vec{V} \perp \vec{r}$ . It is for this reason that $\vec{V}$ is called the tangential velocity. Could there be another? Sure! How about the radial velocity (think of the Doppler radar observations)? That particular component involves a change of the length of $r$ with time. In our example, $r$ = constant (for circular motion) but this needs not be the case always. If $r = r(t)$, then $\frac{d\vec{r}}{dt}$ has another component:

$$= \hat{i}\left[\frac{dr}{dt}\cos(wt)\right] + \hat{j}\left[\frac{dr}{dt}\sin(wt)\right].$$

This part is directed along $\vec{r}$ , as can be seen by comparing this expression to $\vec{r}$ itself. Note that, in this example, our coordinate system was fixed. Suppose now we look at the motion from the particle's point of view, i.e., moving with the particle (just like us standing on the earth surface), then things will be more complicated. We will deal with this in a later chapter.



Let's now look at the acceleration:

$$\vec{a} = \frac{d\vec{V}}{dt} = rw^2[-\hat{i}\cos(wt) - \hat{j}\sin(wt)]$$

$$\vec{a} = w^2(-\vec{r}) \quad (5.14)$$

This acceleration is directed opposite to the position vector (inward toward the axis of rotation) and is called the centripetal acceleration. It is a real acceleration. To realize this acceleration, there has to be a force acting on this circulating object that points to the center of the circle. Think of a ball attached to a string – the string must have tension to full the ball towards the center. It balances the mass times acceleration ( $\vec{F} = m\vec{a}$ ).

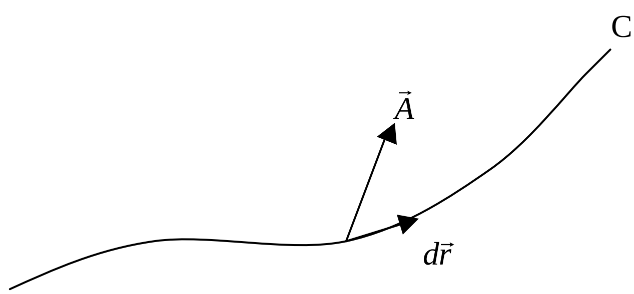
If we put ourselves in the coordinate of the particle (image riding the roller-coaster), we experience an apparent force, called the centrifugal force. It pulls outward but is not real – it only comes into play if we use a rotating coordinate system. We will more on this when we look at the Coriolis force.

## **Integration of Vectors**

The principal integral of vectors is called the line integral. If we have some curve C in space and a vector function $\vec{A}$ defined at points along the curve, then the line integral of $\vec{A}$ along C is given by

$$\int \vec{A} \cdot d\vec{r}$$

where $\vec{r}$ is a position vector. Geometrically, we can imagine curve C divided into small line segements (p.88-89 of Symon) where $d\vec{r}$ is in the direction of the segement → the direction of integration makes a difference! - VERY IMPORTANT!




$d\vec{r}$ is tangential to C

So $\int \vec{A} \cdot d\vec{r}$ = projection of $\vec{A}$ onto the various segment $d\vec{s}$ at points along the curve. Recall that

$$W = \int \vec{F} \cdot d\vec{r}$$

If s = distance measured along the curve, then

$$\int \vec{A} \cdot d\vec{r} = \int A \cos q \, ds$$

If the curve C is closed, then we write

$$\oint \vec{A} \cdot d\vec{r}$$

See p89-90 of Symon for an example of work done by a force on a particle moving along a semi-circle. **<u>Go through it on your own and make sure you understand it!</u>**

There is a special case that we need to consider. Often we can write a vector as the gradient of a potential function:

$$\vec{A} = \nabla f, \quad \nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

where $\phi$ = potential. Think of some example:

$$\vec{F} = -\nabla V \quad \text{(potential function)}$$
$$\vec{g} = -\nabla f \quad \text{(gravitational potential)}$$
$$\vec{E} = -\nabla V \quad \text{(electric potential)}$$

Let's evaluate a line integral in this case between 2 points, $P_1$ and $P_2$:

$$\int_{P_1}^{P_2} \vec{A}\cdot d\vec{r} = \int_{P_1}^{P_2} \nabla f \cdot d\vec{r}$$

Now, recall the chain rule in 3-D:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

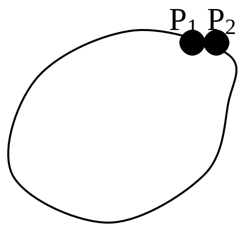
with $d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$ and the definition of $\nabla$ given above, we see that

$$df = \nabla f \cdot d\vec{r}\ .$$

Thus, our integral becomes

$$\int_{P_1}^{P_2} \nabla f \cdot d\vec{r} = \int_{f_{P_1}}^{f_{P_2}} df = f_{P_2} - f_{P_1}\ .$$

If we are dealing with a closed curve, the $P_1 = P_2$,



thus, $\oint \nabla f \cdot d\vec{r} = 0$.

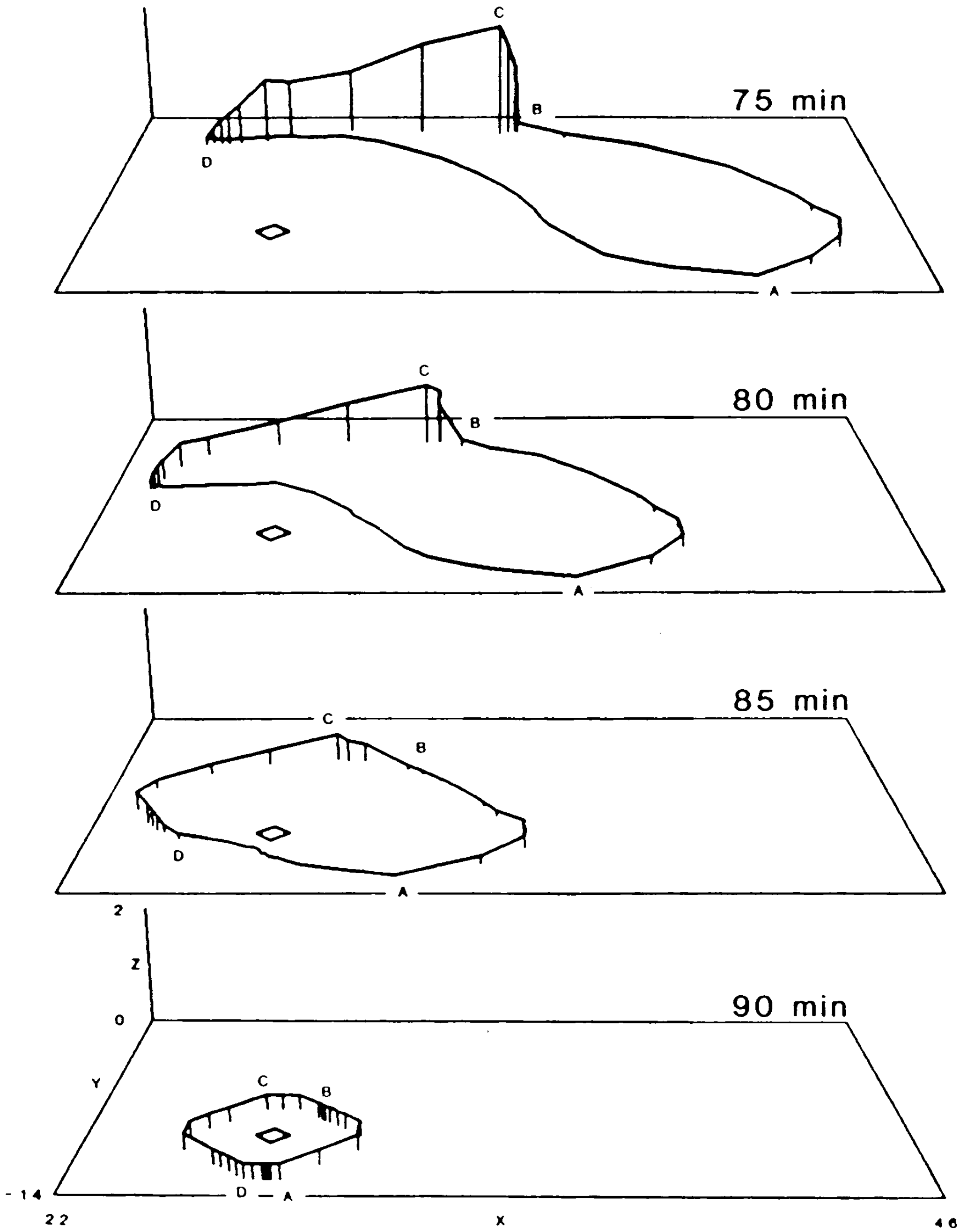
Therefore for any force that can be expressed as a gradient of a scalar function, the total work done alone a closed path is zero! The first part of the sentence actually defines a conservative force!.

**Practical Examples in Meteorology**

Next spring, you will learn about a quantity known as the circulation:

$$C = \int \vec{V} \cdot d\vec{r} \tag{5.15}$$

where $\vec{V}$ = velocity vector. The circulation is a measure of the rotation about a given curve. We use this in storm dynamics to understand how tornado acquires their rotation. The following example.




The above figure shows a perspective plot of the 3D evolution of a material curve in the 3-D tornadic storm simulation by Rotunno and Klemp (1985), who calculated the circulation along the curve.

Rotunno, R., and J. B. Klemp, 1985: On the rotation and propagation of simulated supercell thunderstorms. *J. Atmos. Sci.*, **42**, 271-292.

## 5.3. Gradient, Divergence and Curl (Read p98-102 of Symon)

**Gradient**

Suppose we have a scalar function A that is *A(x,y,z)*. We define the gradient vector as

$$grad(A) = \nabla A \equiv \hat{i}\frac{\partial A}{\partial x} + \hat{j}\frac{\partial A}{\partial y} + \hat{k}\frac{\partial A}{\partial z} \tag{5.16}$$

Note that $\nabla A$ is a vector! Also, one can write $\nabla A$ in other coordinate system, e.g., the spherical coordinate. As we saw earlier, $dA = \nabla A \cdot d\vec{r}$. $\nabla = "del"$.

Think of the gradient of a 1D curve [ y=f(x) ]

**Divergence**

We can use the gradient in a special way – to form the divergence of $\vec{A}$ where $\vec{A}$ is now a vector, $\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z$:

$$\mathrm{Div}(\vec{A}) = \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{5.17}$$

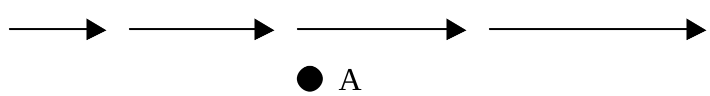
Note that to be "divergence", $\vec{A}$ need not be the velocity. Later we will look at integrals involving the divergence. For now, let's look at it in terms of the wind field:

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad = \text{a scalar quantity}$$

It is the velocity divergence in Cartesian coordinates.

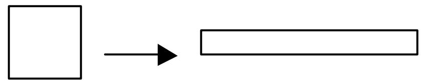
Consider on one dimension:

Divergence:

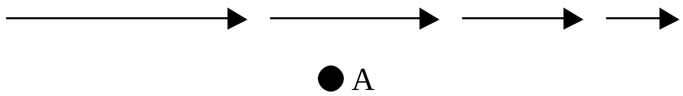



$\nabla \cdot \vec{V} > 0$ is called divergence [ for 1-D, $\frac{\partial u}{\partial x} > 0$ ]

From the figure, there is divergence at point A – one way to tell is to put a deformable box at the point, to see how the box deforms – if it is stretched, there is divergence!



Convergence:




$\nabla \cdot \vec{V} < 0$ is called convergence [ for 1-D, $\frac{\partial u}{\partial x} > 0$ ].
Now, there is convergence at point A. A box placed there will shrink!

If $\nabla \cdot \vec{V} = 0$, a fluid is said to be INCOMPRESSIBLE because based on the mass continuity equation, the density of the fluid will not change in this case.

$\nabla \cdot \vec{V} = 0$ is also called non-divergent. We can consider the atmosphere to be incompressible if we look at shallow depth – on the order of 10 km.

We can use the $\nabla \cdot \vec{V} = 0$ condition to compute the vertical velocity field given the horizontal wind field. This is called the kinetic method for computing w (vertical velocity).

Let's rewrite $\nabla \cdot \vec{V} = 0$ as

$$\frac{\partial w}{\partial z} = -\nabla_h \cdot \vec{V}_h$$

where w is vertical velocity and "h" denotes horizontal.

If we integrate in the vertical direction, we obtain

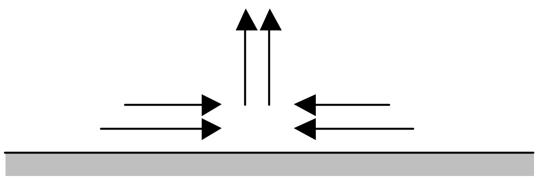
$$\int_{z_1}^{z_2} \frac{\partial w}{\partial z} dz = -\int_{z_1}^{z_2} \nabla_h \cdot \vec{V}_h dz = -\int_{z_1}^{z_2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz$$

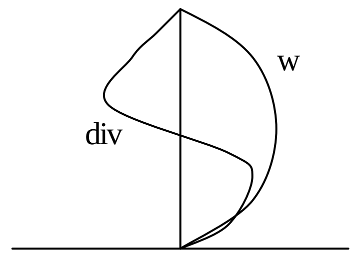
$w(z_2) – w(z_1) = RHS$.

For flat ground, w at z=0 is zero, so if we now the horizontal wind, we can compute the vertical wind w at any height! If $\nabla_h \cdot \vec{V}_h = \text{constant}$, then $w(z) = z\left(-\nabla_h \cdot \vec{V}_h\right)$.

So, w for an incompressible fluid is the integral of the horizontal divergence at all levels below.

In thunderstorms and fronts, the low-level convergence is associated with upward motion:






$\frac{\partial u}{\partial x} < 0 \rightarrow$ convergence

$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x}$, $w=w_{max}$ when $\frac{\partial u}{\partial x} = 0$

Convergence blow → upward motion above. You know this as Dine's compensation in 1004, but now you see it from a mathematical view point. Note that w=0 at z=0 (ground is flat) is called kinematic boundary condition – i.e., the flow cannot penetrate the ground. Keep in mind that w at any level = INTEGRAL of divergence from blow.

Let's now look at the integral of $\nabla \cdot \vec{V}$ over a volume $\Omega$ bounded by a surface S. The so-called "**Gauss ' Divergence Theorem**" says that

$$\iiint_{\Omega} \nabla \cdot \vec{V} d\Omega \equiv \iint_{S} \vec{V} \cdot \vec{n} ds \qquad (5.18)$$

where $\vec{n}$ is the unit vector on the surface pointing outward from the volume.

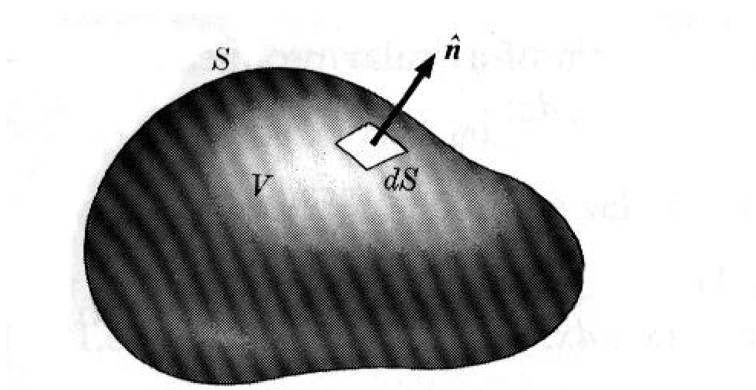
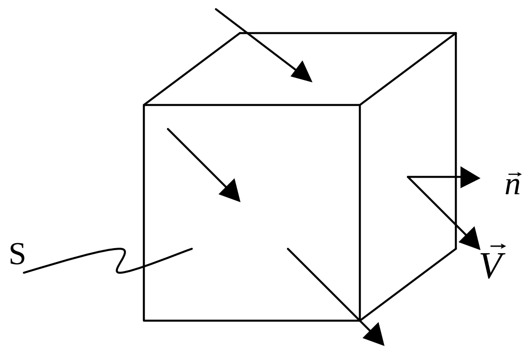



**Fig. 3.24** A volume $V$ bounded by a surface $S$.

Let's look at the special case where this volume is a cubic box with length L on each side. The area of each side is therefore $S=L^2$ and volume of the box is $V=L^3$.

The RHS of (5.18) is the integral of the normal component of $\vec{V}$ (projection of $\vec{V}$ onto $\vec{n}$) over the area S.




For the box face on the right, $\vec{V}\cdot\vec{n}=u_{right}$, for the left face, $\vec{V}\cdot\vec{n}=-u_{left}$ because the outward pointing $\vec{n}$ points in the negative x direction. Similarly for y and z directions. Therefore,

$$\iint_S \vec{V}\cdot\vec{n}\,ds=\int_{z_1}^{z_2}\int_{y_1}^{y_2}(u_{right}-u_{left})dydz+\int_{z_1}^{z_2}\int_{x_1}^{x_2}(v_{back}-v_{front})dxdz+\int_{y_1}^{y_2}\int_{x_1}^{x_2}(w_{top}-w_{bottom})dxdy \qquad (5.19)$$

Clearly, the above represent the net flux through the box surface.

On the LHS, $\nabla\cdot\vec{V}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$,

$$\begin{aligned}\iiint_\Omega \nabla\cdot\vec{V}d\Omega &=\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]dxdydz \\ &=\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2}\frac{\partial u}{\partial x}dxdydz+\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2}\frac{\partial v}{\partial y}dxdydz+\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2}\frac{\partial w}{\partial z}dxdydz\end{aligned} \qquad (5.20)$$

Let's look at the first term on the RHS:

$$
\begin{aligned}
&\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2}\frac{\partial u}{\partial x}dxdydz=\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2}\frac{\partial u}{\partial x}dxdydz=\int_{z_1}^{z_2}\int_{y_1}^{y_2}\left[\int_{u_{left}}^{u_{right}}du\right]dydz\\
&=\int_{z_1}^{z_2}\int_{y_1}^{y_2}(u_{right}-u_{left})dydz\\
&=\int_{z_1}^{z_2}\int_{y_1}^{y_2}u_{right}dydz-\int_{z_1}^{z_2}\int_{y_1}^{y_2}u_{left}dydz\\
&=\text{fluxthrough right boundary - flux through left boundary}
\end{aligned}
\tag{5.21}
$$

The same thing can done for the 2nd and 3rd term on the RHS of (5.20). Therefore, we have proven the Gauss' theorem, $\iiint_\Omega \nabla\cdot\vec{V}d\Omega\equiv\iint_S\vec{V}\cdot\vec{n}ds$, for the cubic volume.

When the normal velocity is constant on each face, we have

$$
\iiint_\Omega \nabla\cdot\vec{V}d\Omega=\iint_S\vec{V}\cdot\vec{n}ds=(u_{right}-u_{left})S+(v_{front}-v_{back})S+(w_{top}-w_{bottom})S
$$

Therefore, the total amount of divergence in the volume is equal to the integral of what passes through the surface of the volume.

**Curl** :

The final vector operator that we will consider is the curl – defined for a vector $\vec{A}$ as

$$
\nabla\times\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix} \quad \text{(= a vector)}
$$

$$
=\hat{i}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)-\hat{j}\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)+\hat{k}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right). \tag{5.22}
$$

If $\vec{A}=\vec{V}$, the vector velocity, then we have what is called the vorticity vector $\vec{w}$ = curl $(\vec{V})$.

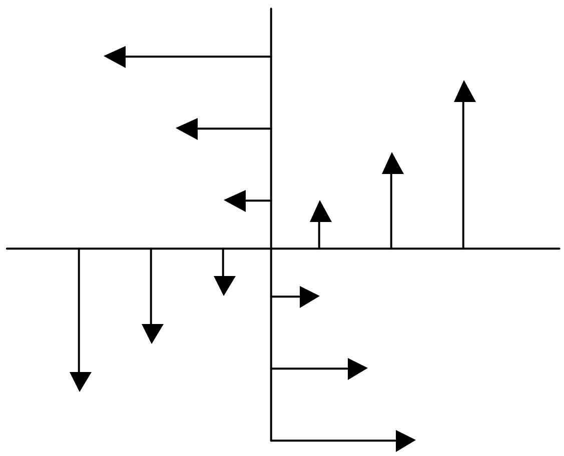
In the atmosphere, all 3 components are important, but the vertical component is most critical:

$$\hat{k}\cdot\left(\nabla\times\vec{V}\right)=\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \tag{5.23}$$

Vorticity describes the microscopic rotation of a particle. You will see in Dynamics I that $\zeta$ = C/A in the fluid as A→0.

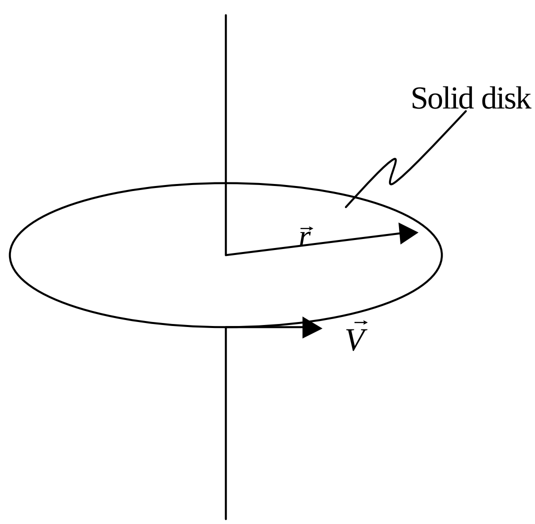
Let's look at a simple example:

u = - ay, v = ax:



Important note: The direction of $\nabla\times\vec{A}$ is not determined by the RH rule – it has 3 components. $\nabla$ is not a vector, but a vector operator.

Let's look at the special example of solid body rotation - we will not spend much time on this in physical mechanics, but it can be relevant to the atmosphere (e.g., tornado core).



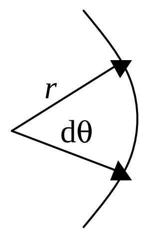

What is the circulation for the circle of radius r?

$$\vec{V} = \vec{\Omega} \times \vec{r} = \text{tangential velocity}$$

(remember that V = ω r, where ω = dθ/dt )

$$C = \int \vec{V} \cdot \overrightarrow{dl} = \int (\vec{\Omega} \times \vec{r}) \cdot \overrightarrow{dl}$$

$$\overrightarrow{dl} = r\,\overrightarrow{d\theta}$$




Therefore

$$C = \int_0^{2\pi} |\vec{\Omega}||\vec{r}|\sin(90^\circ)\, r\ d\theta = 2\pi\ \Omega r^2$$

The area of the disk = π $r^2$, so the vorticity is C/A = 2Ω → vorticity for an object in solid body rotation is twice its angular rotation rate. This we can see from the previous drawing and the definition of vorticity $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$.

The final aspect of $\nabla \times \vec{V}$ to examine is known as Stokes Theorem:

$$\iint (\nabla \times \vec{V}) \cdot \vec{n}\, ds = \int \vec{V} \cdot d\vec{r} \qquad (5.24)$$

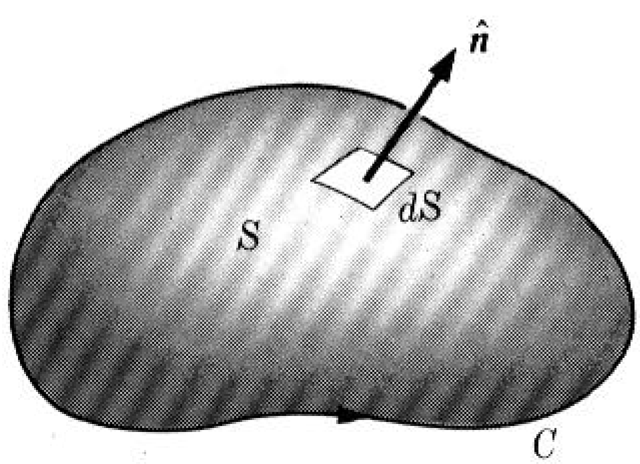



**Fig. 3.25** A surface *S* bounded by a curve *C*.

It says for a given surface enclosed by a curve C, the surface integral of the vorticity, $\nabla \times \vec{V}$, in the direction of the outward normal vector, $\vec{n}$, equals the line integral of velocity along curve C.

We said earlier that you will see in Dynamics I that the vorticity $\zeta$ = C/A in the fluid as A→0, where C is the circulation surrounding an area A. We accept this as true, then we can understand the Stokes Theorem in the following way.

Consider the following 2-D area S, divided into many small 'cells' that contain circulations. When the cells are small enough, then the vorticity associated with each cell

$$\zeta_i = C_i / A_i \quad \text{or} \quad \zeta_i A_i = C_i \tag{5.25}$$

where $A_i$ is the area of the individual cell. Adding both sides of the equation for all cells, we have

$$\sum_i V_i A_i = \sum_i C_i \tag{5.26}$$

We see that in the limit that $A_i$ approaches zero, the LHS of (5.26) is essentially the area integral of vorticity $\zeta$. The RHS is the summation of the circulation in all the cells. Notice that the circulation at the interior boundaries of neighboring cells are of opposite directions therefore the internal circulations cancel each other – only the circulation at the outmost boundaries are left over, which essentially gives the circulation along the curve enclosing the entire area! More vigorous proof of the Stokes Theorem for a general 3-d surface can be found in advanced calculus books, we will not go into detail here.