METR 5303 – Lecture #8

Filtering and Its Response Function

The purpose of this lecture is to introduce the topic of filtering and the use of filter response functions to understand how arithmetic operations affect the spectral character of the data. This will also enable greater understanding of the filtering properties of the Barnes objective analysis scheme.

Purpose of Filtering

The most common use of filtering in objective analysis is to smooth or remove "undesired noise" from a field while retaining "features of interest". Both goals in quotes are subjective and are defined by the user. That is, one should learn how to design a filter to suit their needs.

When the observational errors are not spatially correlated, an objective analysis algorithm should attempt to extract the large-scale signal while suppressing the small-scale observational noise. Thus the objective analysis algorithm acts as a filter as well as an interpolator.

One-dimensional filter

Consider the following one-dimensional, 3-point symmetric filter operator:

$$\bar{\phi}_j = (1 - S)\,\phi_j + S/2\,(\phi_{j+1} + \phi_{j-1}) \qquad (1)$$

for the following grid:

S is the smoothing factor that determines the filter weights

For example, if $S = ½$, the weights are ¼, ½, and ¼ for ϕ_{j-1}, ϕ_j and ϕ_{j+1} respectively,

One can see that a smaller value of *S* gives more weight to the center point.

Eq. (1) represents an arithmetic operation on data, as does any calculation, data analysis or objective analysis procedure, which nearly always <u>changes the spectral character of the data</u>.

To examine the change caused by such operations to individual wavelengths (or frequencies) of the signal, we need to determine the <u>response function</u> *R* of the filter.

In general, R can be expressed as $R = \bar{\phi}(k)/\phi(k)$, $k = 2\pi/L$ (wavenumber)

For example, assume our data can be represented by sinusoidal fields so that

$$\phi(k) = A\, e^{ikx}$$

On a discrete grid, this is written as $\phi_j = A\, e^{ikj\Delta x}$, $j = 1, M$ (2)

Note that the maximum wavelength is $L_{max} = M\Delta x$.

Now substitute eq. (2) into the 2nd term on the RHS of eq. (1):

$$\bar{\phi}_j = (1-S)\phi_j + S/2\,[\,Ae^{ik(j+1)\Delta x} + Ae^{ik(j-1)\Delta x}]$$

$$= (1-S)\phi_j + S/2\,Ae^{ikj\Delta x}[e^{ik\Delta x} + e^{-ik\Delta x}]$$

Since $\phi_j = Ae^{ikj\Delta x}$, and $e^{ik\Delta x} + e^{-ik\Delta x} = 2\cos k\Delta x$ (via Euler's rule),

we have

$$\bar{\phi}_j = \phi_j\,[(1-S) + S\cos k\Delta x] = \phi_j[1 - S(1-\cos k\Delta x)] \qquad (3)$$

or, since $1-\cos x = 2\sin^2 x/2$,

$$\bar{\phi}_j = \phi_j\,[\,1 - 2S\sin^2 kDx/2\,]\ .$$

Therefore, this is in the form $\bar{\phi}_j = R\phi_j$, where

$$R = 1 - 2S\sin^2 p\Delta x/L \qquad (4)$$

Thus R = $\bar{\phi}_j/\phi_j$ is the ratio of the amplitude of the smoothed (new) wave component to the old one.

Note that if $S < 0$, $R > 1$, and we get amplification of the wave - not usually desired.

If $R < 1$, we get damping of the wave component - which causes smoothing.

However, if we get $R < 0$, we get a 180° phase shift - also not desired.

For symmetric filters whose weights sum to one, the mean of the smoothed field approaches that of the original field.

Consider a particular value of S, S = 1/2 :

Eq. (1) becomes $\bar{\phi}_j$ = ¼ $(\phi_{j+1} + \phi_{j-1} + 2\phi_j)$ (5)

And the filter weights, as stated earlier, are ¼, ½, and ¼ for ϕ_{j+1}, ϕ_j and ϕ_{j-1} resp.

This is often called the "1 – 2 – 1 filter" or "hanning".

We will use eq. (3) for R, which becomes

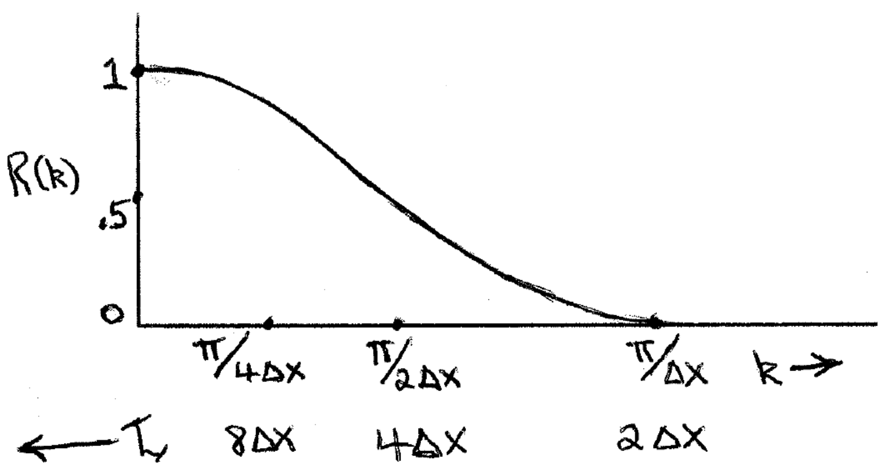
$$R = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\pi\Delta x/L \qquad (6)$$

We can see from eq. (6) that if $L = 2\Delta x$ (the Nyquist wavelength), then $R = 0$.

Also, if $L = 4\Delta x$, $R = 1/2$; $L = 8\Delta x$, $R = 0.8535$; ……..; $L = \infty$, R approaches 1.

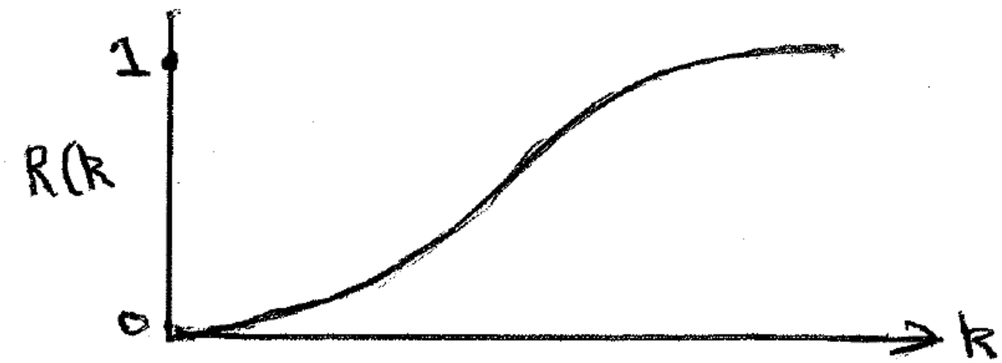
That is, we have zero response at 2Δx, 50% response at 4Δx, etc., while the response for long wavelengths (or frequencies) approaches 100%.

Plotting eq. (6), we get:

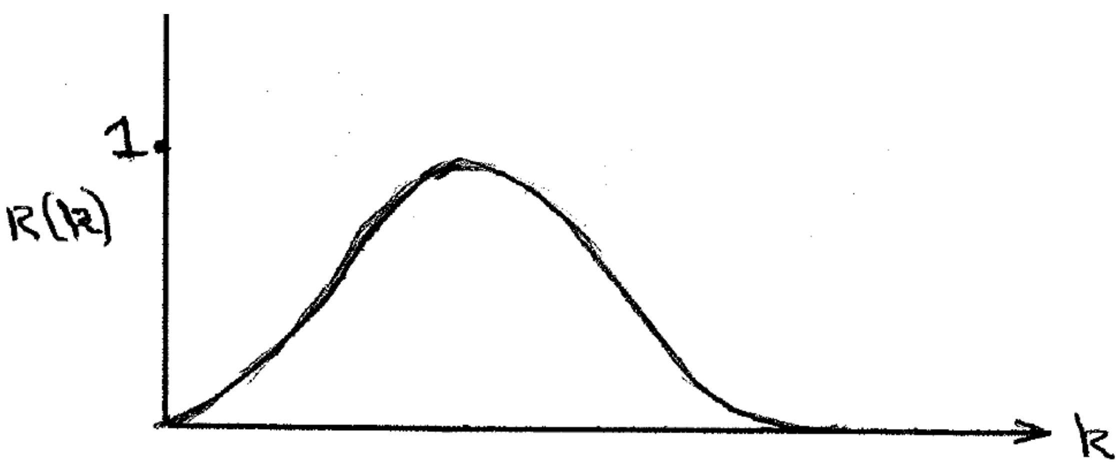


This is an example of a <u>low pass filter</u>; i.e., it "passes" (does not greatly affect) low wavenumbers (large wavelengths), but smoothes small-scale features.

We could also design <u>high-pass filters</u>:



as well as <u>band-pass filters</u>:



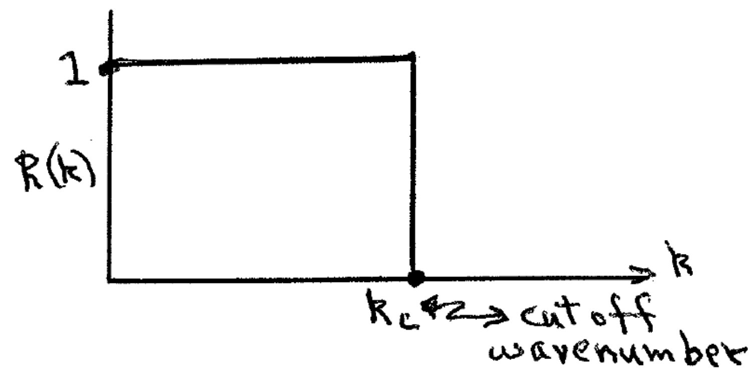
If we apply the same filter m times, the response is, using eq. (4), $S = ½$ and the trig. identity $\sin^2 x + \cos^2 x = 1$,

$$R(k)^m = [\cos^2 \pi\Delta x/L]^m$$

Thus if, e.g., $L = 8\Delta x$: $R = .8535$ after 1 pass
$R = .53$ after 4 passes (or applications)
$R = .08$ after 16 passes

Therefore, low-pass filters can rapidly damp out even long waves with repeated applications.

The "ideal" response function is one that has $R = 1$ for the scales one desires to retain, and $R = 0$ for the undesired scales, as shown to the right. The design of filters that has this and/or other desired properties is a major field in some disciplines (e.g., in EE).



To create a simple example of filter design, we assume we know that the response function of a filter is eq. (4):

$$R = 1 - 2S \sin^2 \pi\Delta x/L,$$

and we desire the response R to be zero when $L = 2\Delta x$.

Setting R to 0, we have

$$0 = 1 - 2S \sin^2 \pi\Delta x/2\Delta x = 1 - 2S, \text{ and thus } S = ½ \text{ accomplishes this.}$$

If we desire two or more different filters to be used in sequence, the total response $R_n(k)$ is:

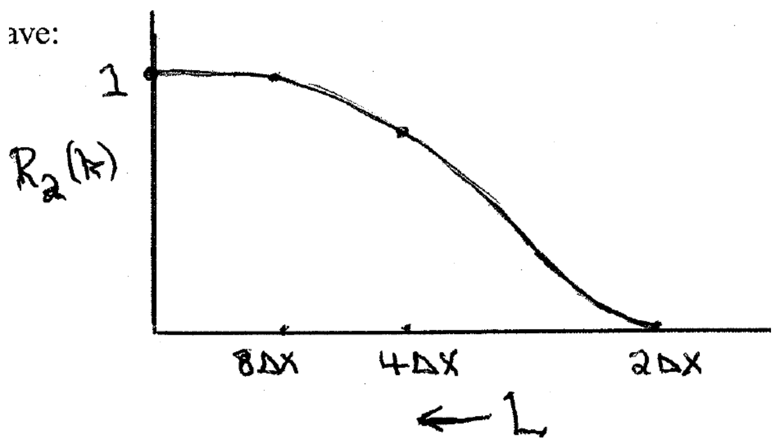
$R_n(k) = R_1 R_2 \ldots\ldots R_n$. If all the filters are symmetric such that we can

use eq. (4), we have $R_n(k) = \Pi\ (1 - 2S_i \sin^2 k\Delta x/2)$

For example, if $S_i = 1/2$, followed by a filter operation with S = -1/2 , we obtain a five-point symmetric filter with

$$R_2(k) = 1 - \sin^4 k\Delta x/2 \qquad (7)$$

Plotting this, we have:



Thus, with just 2 passes, we have a "sharper" response. Higher values of n with judicious choices for S can make the response even sharper, but it takes more weights (i.e. – it requires more data values per point) to accomplish this (which effectively shrinks your smoothed domain). Thus compromises are always necessary.

Class exercise: Show that the filter weights for the above n = 2 case are -1/16, 1/4, 5/8, 1/4, and -1/16 for the ϕ_{j-2}, ϕ_{j-1}, ϕ_j, ϕ_{j+1} and ϕ_{j+2} points resp. Note that these weights sum to 1.

In general, a sharper filter can be designed by using more weights, but this comes at the expense of needing more surrounding data. Thus, e.g., one is unable to use the filter near the edges of a domain.

Note that some values of S may lead to poor results. For example, if S = 1, such that the filter becomes

$$\bar{\phi}_j = \tfrac{1}{2} (\phi_{j+1} + \phi_{j-1})$$

Class exercise: Show that the response function for this filter produces a 180^O phase change for $L = 2\Delta x$ with no reduction in amplitude.

If S = -1/2, the resulting filter will cause amplification for some values of k. This has been called a "de-smoother", and was actually used by NMC (former NCEP) in the early days of NWP to post-process the forecasts.

<u>2-D Filtering</u>

To filter a two-dimensional grid, we commonly apply our one-dimensional filter in both directions to smooth the fields. If we do this with the 1 – 2 – 1 filter, the weights are

$$\begin{matrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{matrix} \quad , \text{ all divided by 16.} \qquad (\textbf{Show this.})$$

The 2-D response function for this is

$$R(k, l) \quad = \quad (1 - \sin^2 k\Delta x/2)\,(1 - \sin^2 l\Delta y/2)$$

This is a 9-point filter, which can be written out as $\bar{\phi}_{i,j}$ = 9 values of $\phi_{i,j}$ and $\phi_{i\pm 1,\, j\pm 1}$ using the weights given above.

Note: Much of this material came from Haltiner and Williams NWP book, p. 392-398.