METR 5303 – Lecture #2

Objective Analysis Fundamentals

Definition: **Objective Analysis -** A procedure for obtaining estimates of irregularly-spaced field variables to points on a regular grid.

This is usually in 2-D or 3-D space but could be in time.

Objective analysis usually has additional, simultaneously-desired goals:

1. Detection and suppression of **noise** in the data

We define **noise** as:

(a) Measurement errors

- (i) Instrument errors
- (ii) "gross errors" (e.g., miscoding; transmission errors, etc.)
- (iii) Representativeness (is point measurement representative of a desired area or volume?)

(b) Features whose scale is less than twice the sampling interval - i.e. – the **Nyquist frequency or wavelength**

(c) Interpolation errors

Thus, we may desire to filter out this noise. Also may filter to retain only scales appropriate for the problem.

2. For **multivariate analysis** (more than one variable analyzed simultaneously), we may desire that the <u>fields are dynamically consistent</u>. For example,

(a) $\vec{\mathbf{V}}$ and ϕ should be in <u>quasi-geostrophic balance</u> for larger-scale motions

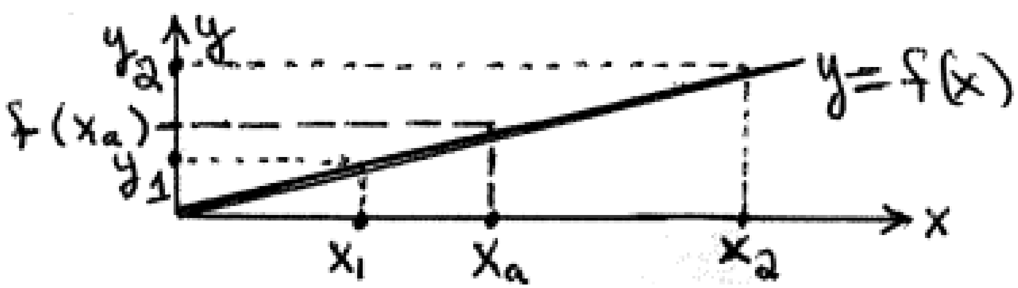
(b) Mass continuity should be preserved

(c) Fields should be in balance with the model equations (goal of 4DDA)

<u>Interpolation Basics</u>

1. **<u>Linear Interpolation</u>**

Consider the following “observations” y_1 , y_2 of the 1-D function $y = f(x)$ at points x_1 , x_2 :



What is the value of $f(x)$ at $x = x_a$?

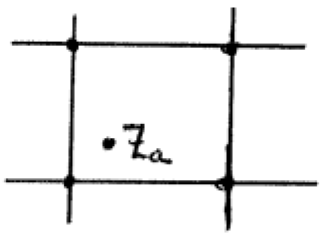
The linear interpolation estimate is:

$$f(x_a) = f(x_1) + \frac{x_a - x_1}{x_2 - x_1} \; f(x_2) - f(x_1) \quad . \qquad (1)$$

Note: One should always examine formulas to see how they work. Here, we see if x_a is close to x_1 , the 2nd term is small and the answer is close to $f(x_1)$. If x_a is close to x_2 , the coefficient is close to 1, the two $f(x_1)$'s cancel, and the answer is close to $f(x_2)$, as desired.

Class exercise: Develop a formula for 2-D linear interpolation – **Bilinear interpolation**

Use the grid shown to right and assume 4 height observations as shown at grid intersections. Develop a formula to obtain $Z_a(x, y)$. We will use this formula later.



Note: Equation (1) can also be written in the form:

$$f(x_a) = c_0 + c_1 x \qquad (2)$$

where

$$c_0 = \frac{x_2}{x_2 - x_1} f(x_1) - \frac{x_1}{x_2 - x_1} f(x_2),$$
$$c_1 = \frac{-1}{x_2 - x_1} f(x_1) - \frac{1}{x_2 - x_1} f(x_2). \qquad (3)$$

Class exercise: Prove this.

In eq. (2), 1 and *x* are basis functions for a linear model - part of a set of polynomial basis functions 1, x, x^2 , x^3 , etc.

As another example, sin x, cos x are the basis functions for a Fourier Series

Thus, in (2), c_0 , c_1 are estimated parameters or coefficients that need to be determined. Equation (3) expressed those coefficients in terms of known values.

The general form for eq. (3) is:

$$c_i = \sum_{j=1}^{2} \left[h_{ij} f(x_j) \right] \tag{4}$$

So, in eqs (2) and (3), i = 0 and 1, and the coefficients h_{ij} are

$$h_{01} = \frac{x_2}{x_2 - x_1}, \quad h_{02} = -\frac{x_1}{x_2 - x_1}, \text{etc}$$

The use of eqs (1) or (2) within $x_1 < x < x_2$ (the analysis domain) is interpolation

The use of eqs (1) or (2) outside $x_1 < x < x_2$ is extrapolation - avoid at all costs!

Note: We need to realize that the function model (here linear) may be used incorrectly.

Let $\mu(x)$ be the expected value of $f(x_a)$, i.e. , $< f(x_a) >$. If eq (2) is *not* the "true model" for f(x), (that is, the actual field is not linear), then

$$\langle f(x_a) - \mu(x) \rangle \neq 0 \quad \text{over the analysis domain, and}$$

we will have biased estimates of the signal.

Note that $\langle \ \rangle$ here represents the ensemble average over all possible realizations.

Therefore, we have 2 sources of error to consider:

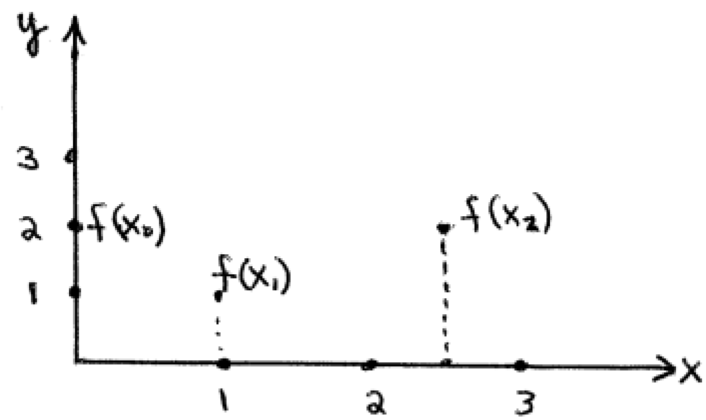
(a) Observational error
(b) Mis-specification of the function model

It is often not possible to tell them apart; need to minimize both.

2. Quadratic Interpolation

Consider the following diagram:

$x_0 = 0$
$x_1 = 1$
$x_2 = 5/2$



Let $y = f(x)$ be sampled at 3 points: [x = 0, 1, 5/2]. Assume error-free observations

or sample values at these points: [y = 2, 1, 2] or f(0) = 2; f(1) = 1; f(5/2) = 2.

Problem: How do we use the sample values (observations) to estimate *y* for other values of *x*?

For example, here we will obtain a value for f(2).

Solution: If *f*(*x*) is suitable "well-behaved" (e.g., all derivatives exist, etc.), *f*(*x*) can be represented by a power series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x_n \frac{d^n f(0)}{dx^n} \qquad (1)$$

[Eq. (1) is a Taylor Series expansion about x = 0, or a Maclaurin Series.]

If we truncate eq. (1) at $n = 2$, we have the estimate

$$\tilde{f}(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) \qquad (2)$$

Thus $\tilde{f}$ (x) will approach *f*(*x*) according to:

(a) How close *x* is to the origin
(b) Smallness of the higher-order derivatives

We now rewrite (2) as

$$\tilde{f}(\mathrm{x}) = a_0 + a_1 x + a_2 x^2 \qquad (3)$$

which is a quadratic interpolating polynomial.

We need to determine a_0 , a_1 , a_2. We shall use the method of undetermined coefficients.

From (3), we can write:

$\tilde{f}$ (0) = a_0 [= f(0) since we assumed no error]

$\tilde{f}$ (1) = $a_0 + a_1 + a_2$ [= f(1)]

$\tilde{f}$ (5/2) = $a_0 + 5/2a_1 + 25/4a_2$ [= f(5/2)]

or, in matrix form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 5/2 & 25/4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) \\ f(2) \end{bmatrix} \tag{4}$$

or

$$\mathbf{CA = F}$$

We can solve eq. (4) using Cramer's Rule.

Class exercise: Do this! Verify that $a_0 = 2$, $a_1 = -5/3$, $a_2 = 2/3$ so that eq. (3) is

$$f(x) = 2 - \frac{5}{2}x + \frac{2}{3}x^2 \; , \tag{5}$$

and thus $f(2) = 1\ 1/3$.

Now write eq. (3) in linear operator form:

$$f(x) = \sum_{k=0}^{2} c_k\, f_0(x_k)\,. \qquad (6)$$

We note that (6) is a linear combination of observed values - which are not linear in x.

The coefficients are given by

$$\begin{aligned} c_0 &= 1 - \frac{21}{15}x + \frac{6}{15}x^2, \\ c_1 &= \frac{25}{15}x - \frac{10}{15}x^2, \\ c_2 &= -\frac{4}{15}x + \frac{4}{15}x^2. \end{aligned} \qquad (7)$$

Class Exercise: Verify that (6) and (7) are equivalent to (5), and that f(2) = 1 1/3.

Note #1: Since $a_2 \neq 0$ in eq. (3), *and* the sample values were assumed to have no error, f(x) is not linear. But, if sample values did have errors (i.e. - not exactly = to 2, 1, 2), then the fact that $a_2 \neq 0$ does *not* mean that the true function can not be linear.

Note #2: $f(x)$ could also be a cubic or higher order function, but we can only fit a quadratic function (or less) to 3 sample points. E.g., if $f(x)$ were cubic, we would need at least 4 independent sample points to determine the coefficients of the function.

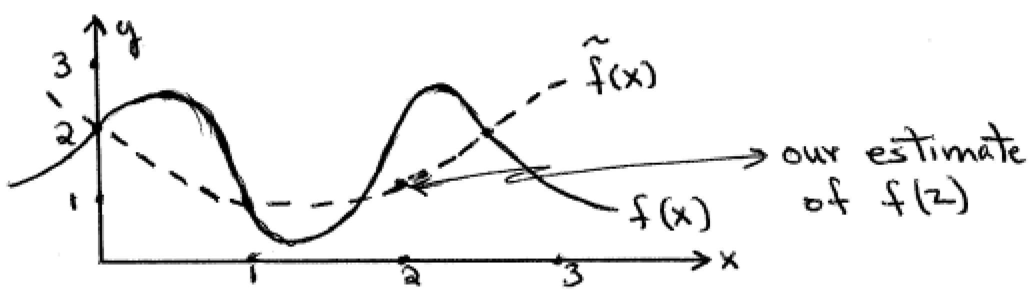
Thus, recall the important rule that "*the degree of the interpolating polynomial is one less than the number of independent sample points*".

Suppose we had 27 sample points. Would you want to fit a 26^{th} order polynomial to the data?

Answer is No! One would probably be fitting noise in the data (overfitting) and the function would be overly "loopy". We will come back to this.

Thus we usually will choose to solve over-determined analysis problems, where the number of points is greater than the order of the polynomial. The least squares technique is one of these approaches.

Note #3: $f(x)$ in general will *not* be quadratic. For example, suppose the true field is the solid line in the diagram below, and the dashed line is what we estimated with eq. (5):



Within the sampling interval – interpolation - our estimates are fair to good.

Outside the sampling interval - extrapolation - our estimate are terrible.

[Expressions for these errors can be found in numerical analysis textbooks.]

So, we have the following error trade-offs:

Low-order polynomials: Smooth fields, but perhaps a poor fit

High-order polynomial: More information (smaller scales) can be fit, but may be fitting noise in the data.

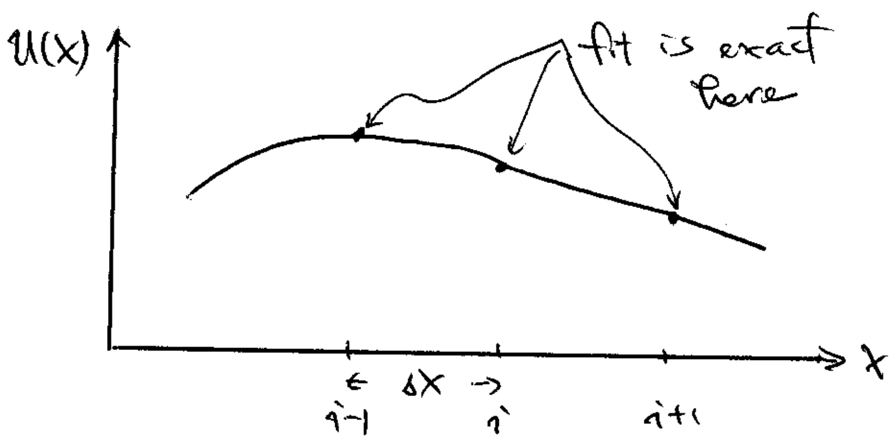
The derivatives of the function may be noisy.

Read Daley, p. 34-39. Review least squares.

Polynomial Fitting: Lecture notes from CFD class:

This is the second, most general method for generating finite difference expression. Here, we assume that the solution to the PDE can be approximated by a polynomial, and that the values at the mesh points are exact. We thus differentiate the polynomial to obtain expression for various derivatives.

Assume that $u(x) = a\,x^2 + b\,x + c$:



Goal: Find a, b and c. Note that the grid spacing need not be uniform.

Applying the polynomial to those three points gives

$$u_{i-1} = a\,x^2_{i-1} + b\,x_{i-1} + c$$

$$u_i = a\,x^2_i + b\,x_i + c$$

$$u_{i+1} = a\,x^2_{i+1} + b\,x_{i+1} + c$$

Solve for *a, b, c*, we obtain

$$u(x) = u_{i-1}\left[\frac{(x-x_i)(x-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})}\right] + u_i\left[\frac{(x-x_{i-1})(x-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})}\right]$$
$$+\, u_{i+1}\left[\frac{(x-x_i)(x-x_{i-1})}{(x_{i+1}-x_i)(x_{i+1}-x_{i-1})}\right]$$

Note: $u(x_i) = u_i$ for *i*, *i*+1, *i*-1 (verify yourself).

The above formula is often called a Lagrange Interpolation Polynomial and can be generalized to any order.