Simple Example of Optimal Interpolation (OI)

The matrix form of the OI equation,

W (HBH^T+R) = BH^T   5.4.31 Kalnay

x$_a$ = x$_b$ + W[y$_o$ –H(x$_b$)]   5.4.26 - Kalnay

can be written, for a particular grid point g influenced by p observations as:

$$x_g^a = x_g^b + \sum_{j=1}^{p} w_{gj} \delta y_j \ , \ \delta y_j = y_o - y_b = y_o - H(x_b) \qquad (1)$$

$$\sum_{j=1}^{p} w_{gj}(b_{jk} + r_{jk}) = b_{gk} \ , \quad k=1,2,\ldots.., \text{p} \qquad (2)$$

b, *r* are background and observation error covariances, respectively.

b_{jk} – between obsevations

b_{gk} = between grid point and observations

Example 1

Single observations p = 1, b_{jk} = 1

(1)→ $x_g^a = x_g^b + w_{g1}(y_o - y_b)$

(2) → $w_{g1}(1+r_{11}) = b_{g1}$

So $$w_{g1} = \frac{b_{g1}}{1+r_{11}}.$$

Given that $w_{g1} \propto b_{g1}$ → larger weight and correction for large error correlation in the background between grid point and observation locations.

And, w_{g1} is inversely proportional to $1+r_{11}$ → large obs error causes smaller weight and correction.

Example 1(a)

If ob is at the grid point, $y_b = x_b$

$$b_{g1} = 1, \; w_1 = 1/(1+r_{11})$$

If no obs eror, $r_{11} = 0$, and

$$x_g^a = x_g^b + (y_o - y_b) = y_o$$ as is required.

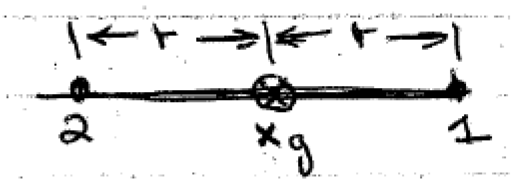
Example 1(b)

If y_o is very far from x_g, b_{g1} → 0, w_{g1} → 0, then

$$x_g^a = x_g^b$$ - no correction by the observation.

Example 2

2 observations, symmetric about $x_{g.}$ Assume no obs error: $r_{jk} = 0$.



Let b be modeled by $b(r) = \exp(-d\, r^2)$ (3)

Weight equation (2) becomes

$$\sum_{j=1}^{2} w_{gj} b_{jk} = b_{gk} \ , k = 1, 2$$

k = 1: $b_{11}\, w_1 + b_{12}\, w_2 = b_{g1}$

k = 2: $b_{21}\, w_1 + b_{22}\, w_2 = b_{g2}$ (4)

By definition, $b_{11} = b_{22} = 1$, $b_{g1} = b_{g2} = b(r)$

And $b_{21} = b_{12} = b(2r) = \exp(-d\, 4r^2) = b(r)^4$

(4) → $w_1 + b^4\, w_2 = b$

$$b^4 w_1 + w_2 = b$$

In this symmetric case, $w_1 = w_2$, so $w_1 = w_2 = b/(1+b^4)$.

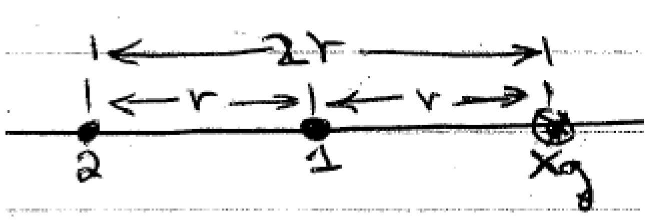
Assume=, for our particular r_1, $b(r)$=0.5, $w_1 = w_2$ =0.47.

If, e.g., background height = 5500 and observation increments are 10, -20, the analysis equation (1) becomes

$$x_g^a = 5500 + 0.47(10) + 0.47\,(-20) = 5495.$$

Example 3

2 stations, asymmetric



Show that equations (4), with $b(r)$ = 0.5, become

$w_1 = b(1+b^2) = 0.625$

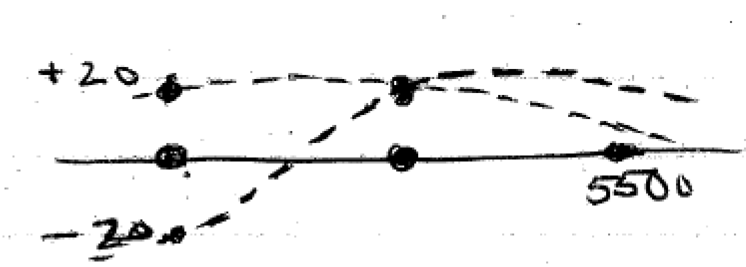
$w_2 = - b^2 = - 0.25$

therefore negative weight can occur.

and, if $\delta y_1 = 20$, $\delta y_2=20$, $x_g^b = 5500$, then $x_g^a = 5507.5$ m.

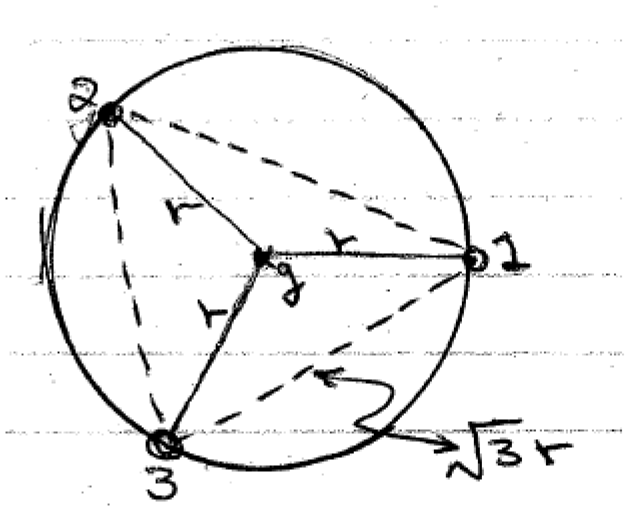
If $\delta y_2 = -20$, $x_g^a = 5517.5$ m,

interpret:



Example 4

3 equi-distant stations, no error.



Euation (2) yields, for p = 3

k = 1: $b_{11}\, w_1 + b_{12}\, w_2 + b_{13}\, w_3 = b_{g1}$

k = 2: $b_{21}\, w_1 + b_{22}\, w_2 + b_{23}\, w_3 = b_{g2}$ (5)

k = 3: $b_{31}\, w_1 + b_{32}\, w_2 + b_{33}\, w_3 = b_{g3}$

12 covariances:

$b_{11} = b_{22} = b_{33} = 1$

$b_{g1} = b_{g2} = b_{g3} = b(r)$

$b_{12} = b_{21} = b_{13} = b_{31} = b_{23} = b_{13} = b_{32} = b(\sqrt{3}\ r) = b^3$.

Since symmetric, $w_1 = w_2 = w_3$,

so, using 1st equation in (5)

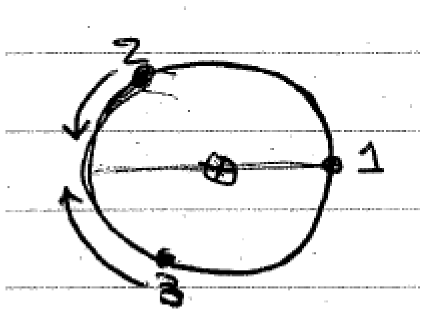
$$w_1 + b^3 w_1 + b^3 w_1 = b_g$$

or $w_1 (1 + 2b^3) = b$ or $w_1 = b/(1 + 2b^3) = w_2 = w_3$.

For $b(r) = 0.5$, $w = 0.40$ (therefore sum of w > 1) and can compute x_g^a.

Example 5.

Same as (4) except the stations are asymetric. E.g, move stations 2 and 3 towards each other.



We will consider case where obs 2 and 3 become coincident opposite to 1.

This cannot be solved as a matrix system since 2 rows will be the same, causing the matrix to be singular. We can solve there by assuming $w_2 = w_3$ and use only 2 equations of (5).

Here $b_{23} = b_{32} = 1$

$$b_{12} = b_{21} = b_{13} = b_{31} = b(2r) \rightarrow b^4.$$

Hence (5) $\rightarrow$ $w_1 + 2b^4 w_2 = b$

$$b_4 w_1 + 2w_2 = b$$

Solving, $w_1 = b/(1+b^4)$, $w_2 = b/(2(b^4+1))$.

Now use (3) $b = \exp(-dr^2)$, choose d = 1x10^{-1} km^{-2} (~ $L = 10^3$ km),

and r = 500 km, then $b = 0.7788$.

So, for example 4, $w_1 = w_2 = w_3 = 0.4005$.

Expected analysis error variance = 0.06435 (not shown).

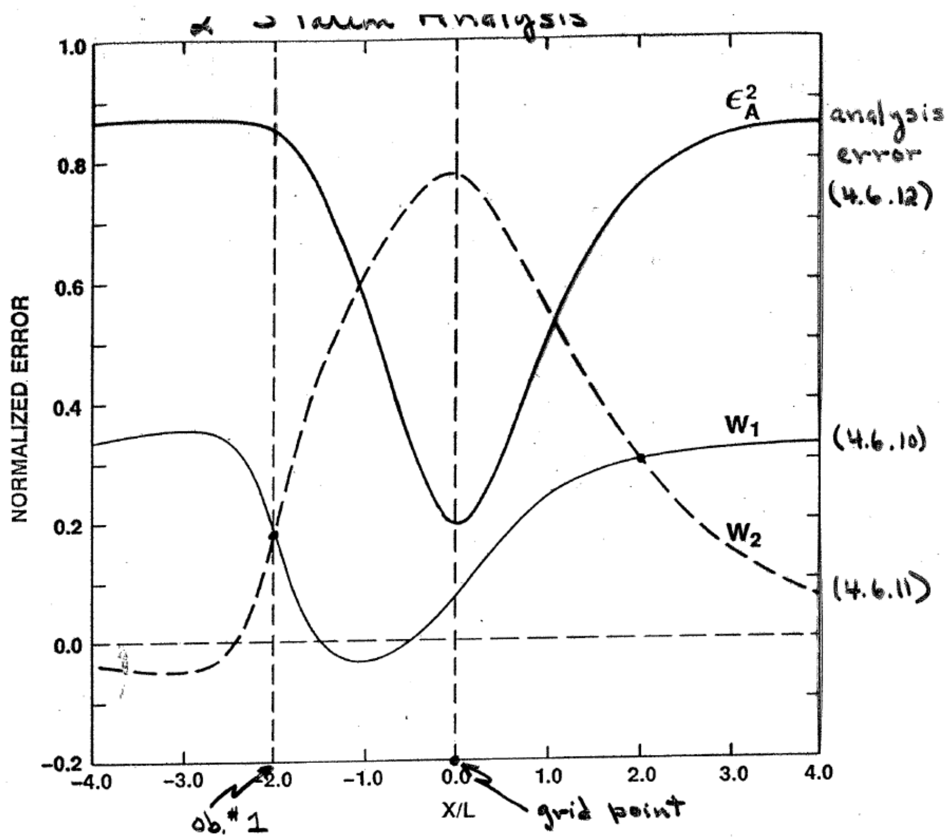
For example 5, $w_1 = 0.56935$ (increases)

$w_2 = w_3 = 0.2847$ (decreases)

$\varepsilon_a^2 = 0.1132$ (increases).

Interpretation: SCM would keep the three weights the same for all positions on the circle.

See how relative positions of observations affects analysis. Figure 4.7 from Daley.



7 A posteriori weights W_1 and W_2 and normalized expected analysis error variance ε_A^2 for the analysis gridpoint at $x=0$, observation 1 at $x=-2.0$ and the position of observation 2 varying between $x=\pm\infty$.

$\varepsilon_1^2 = \varepsilon_2^2 = .25$

(normalized observation error variance)

$$\rho_B(x) = \left(1+\frac{|x|}{L}\right)\exp\left(-\frac{|x|}{L}\right)$$