CARR
FALL, 20

# Simple Examples of O.I.

The matrix form of the OI equations,

$$W(HBH^T + R) = BH^T \qquad 5.4.31 \text{ - Kalnay}$$

$$x_a = x_b + W[y_o - H(x_b)] \qquad 5.4.26 \text{ - "}$$

Can be written, for a particular grid point $g$ influenced by $p$ observations as:

(1) $$x_g^a = x_g^b + \sum_{j=1}^{p} W_{gj}\,\delta y_j \qquad 5.4.33 \text{ "}$$

$$\boxed{\delta y_j = y_o - y_b} \; H(x_b)$$

(2) $$\sum_{j=1}^{p} W_{gj}(b_{jk} + r_{jk}) = b_{gk}, \qquad k = 1, 2 \cdots p$$

5.4.34

$b, r$ are background + observational error covariances.

$b_{jk}$ - between observations

$b_{gk}$ - " grid point + observations

<u>Example 1</u> Single ob. $p=1$; $b_{jk}=1$

(1) $\Rightarrow$ $x_g^a = x_g^b + W_{g1}(y_o - y_b)$

(2) $\Rightarrow$ $W_{g1}(1 + r_{11}) = b_{g1}$

so $W_{g1} = \dfrac{b_{g1}}{(1 + r_{11})}$ $\therefore$ $W_{g1} \propto b_{g1}$ - larger weight for highly correlated error and $w_{g1}$ inversely $\propto$ $r_{11}$ $\therefore$ large obs error causes smaller weight

Ex. 1(a) If ob. at grid point, $y_b = x_b$

$$b_{g1} = 1, \qquad W_1 = \frac{1}{1+r_{11}}$$

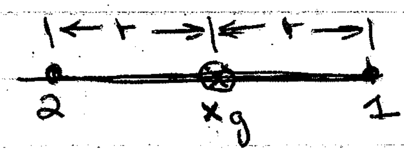
if no obs. error, $r_{11} = 0$,

and $x_g^a = x_g^b + (y_0 - y_b) = y_0$ as is required.

Ex. 1(b) If $y_0$ very far from $x_g$, $b_{g1} \to 0$, $W_{g1} \to 0$

and $x_g^a = x_g^b$ – The background field.

Example 2. 2 obs., symmetric about $x_g$




assume no obs. errors: $r_{jk} = 0$

Let B be modeled by $\boxed{b(r) = e^{-dr^2}}$ (3)

weight eq. (2) becomes

$$\sum_{j=1}^{2} w_{gj} (b_{jk}) = b_{gk} \qquad k = 1, 2$$

$$\left.\begin{aligned} k = 1:&\quad b_{11} W_1 + b_{12} W_2 = b_{g1} \\ k = 2:&\quad b_{21} W_1 + b_{22} W_2 = b_{g2} \end{aligned}\right\} (4)$$

By def., $b_{11} = b_{22} = 1$ ; $b_{g1} = b_{g2} = b(r)$

and $b_{21} = b_{12} = b(2r) = e^{-d(2r)^2} =$

$$= e^{-4\overset{d}{\cancel{b}}r^2} = [b(r)]^4$$

$$\therefore (4) \Rightarrow \quad w_1 + b^4 w_2 = b \qquad (b \equiv b(r))$$

$$b^4 w_1 + w_2 = b$$

In This symmetric case, $w_1 = w_2$

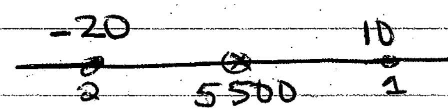
$$\text{so} \quad w_1 = w_2 = \frac{b}{1+b^4}$$

Assume, for our particular $r_i$, $b(r) = .5$ [from (3)]

~~∴ analysis eq. (1) becomes~~

$$\cancel{x_g^a =} \quad w_1 = w_2 = .47$$

If, e.g., background ht = 5500 and observation increments are 10, −20 :



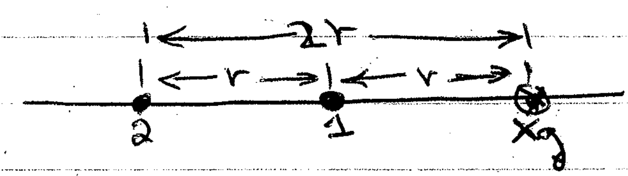

Then analysis eq. (1) becomes

$$x_g^a = 5500 + .47(10) + .47(-20) = 5495\,\text{m}$$

<u>Example 3</u>

2 stations, asymmetrical

<u>show</u> that eqs (4), with $b(r) = .5$, Become




$$w_1 = b(1+b^2) = .625$$

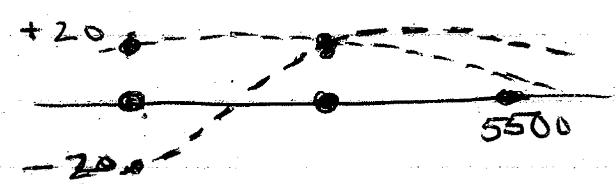
$$w_2 = -b^2 = -.25 \qquad \therefore \text{neg. weights can occur}$$

and, if $\delta y_1 = 20$, $\delta y_2 = 20$, $x_g^b = 5500$

Then $x_g^a = 5507.5$ m

If $\delta y_2 = -20$, $x_g^a = 5517.5$ m

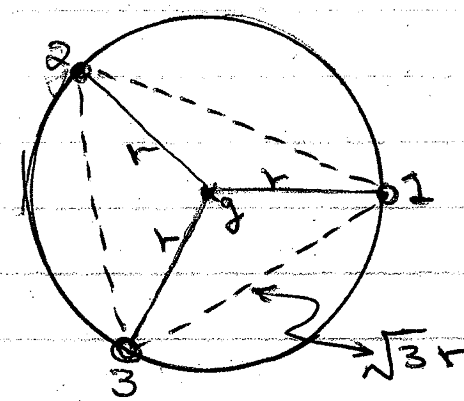
Interpret :




Example 4.

3 equidistant stations:

- no errors




Eq. (2) yields, for $p = 3$

$k=1$: $b_{11}W_1 + b_{12}W_2 + b_{13}W_3 = b_{g1}$

$k=2$ $b_{21}W_1 + b_{22}W_2 + b_{23}W_3 = b_{g2}$

$k=3$ $b_{31}W_1 + b_{32}W_2 + b_{33}W_3 = b_{g3}$ $\Big\}$ (5)

12 covariances : $b_{11} = b_{22} = b_{33} = 1$

$b_{g1} = b_{g2} = b_{g3} = b(r)$ via (3)

$b_{12} = b_{21} = b_{13} = b_{31} = b_{23} = b_{32} = b(\sqrt{3}r)$
$= b^3$

since symmetric, $W_1 = W_2 = W_3$,

so, using $1^{st}$ eq. in (5)

$W_1 + b^3 W_1 + b^3 W_1 = b_g$ or

$$W_1 (1+2b^3) = b \quad \text{or} \quad \boxed{W_1 = \frac{b}{1+2b^3}} = W_2 = W_3$$

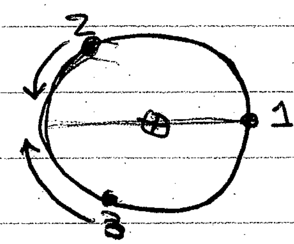
for $b(r) = .5$, $W = .40$ $\quad$ ($\therefore \sum W_j > 1$)

and can compute $x^a_g$ equation

Example 5

Same as (4) except the stations are asymmetric

e.g., move stations 2 and 3 toward each other and re-compute eq. (5). We will consider case where obs. #2 + 3 become coincident - opp. to #1.




[This can not be solved as a matrix system since 2 rows will be the same, causing the matrix to be singular. We can solve here by assuming $W_2 = W_3$ and use only 2 equations of (5)

Here, $b_{32} = b_{23} = 1$

$$b_{12} = b_{21} = b_{13} = b_{31} = b(2r) \Rightarrow b^4$$

$\therefore$ (5) $\Rightarrow$

$$W_1 + 2b^4 W_2 = b$$

$$b^4 W_1 + 2W_2 = b$$

solving, $\boxed{W_1 = \frac{b}{1+b^4}}$, $\boxed{W_2 = \frac{b}{2(b^4+1)}}$

now use (3) $b = e^{-dr^2}$, choose $d = 1\times10^{-6}\,\text{km}^{-2}$

($\sim L = 10^3$ km)

and $r = 500$ km

$\therefore\ b = .7788$

So, for Ex. 4, $W_1 = W_2 = W_3 = .4005$

~~For Ex. 5~~ Expected Analysis Error Variance $= .06435$ (not shown)

For Ex. 5, $W_1 = .56935$ (increases)

$W_2 = W_3 = .2847$ (decreases)

$\varepsilon_A^2 = .1132$ (increases)

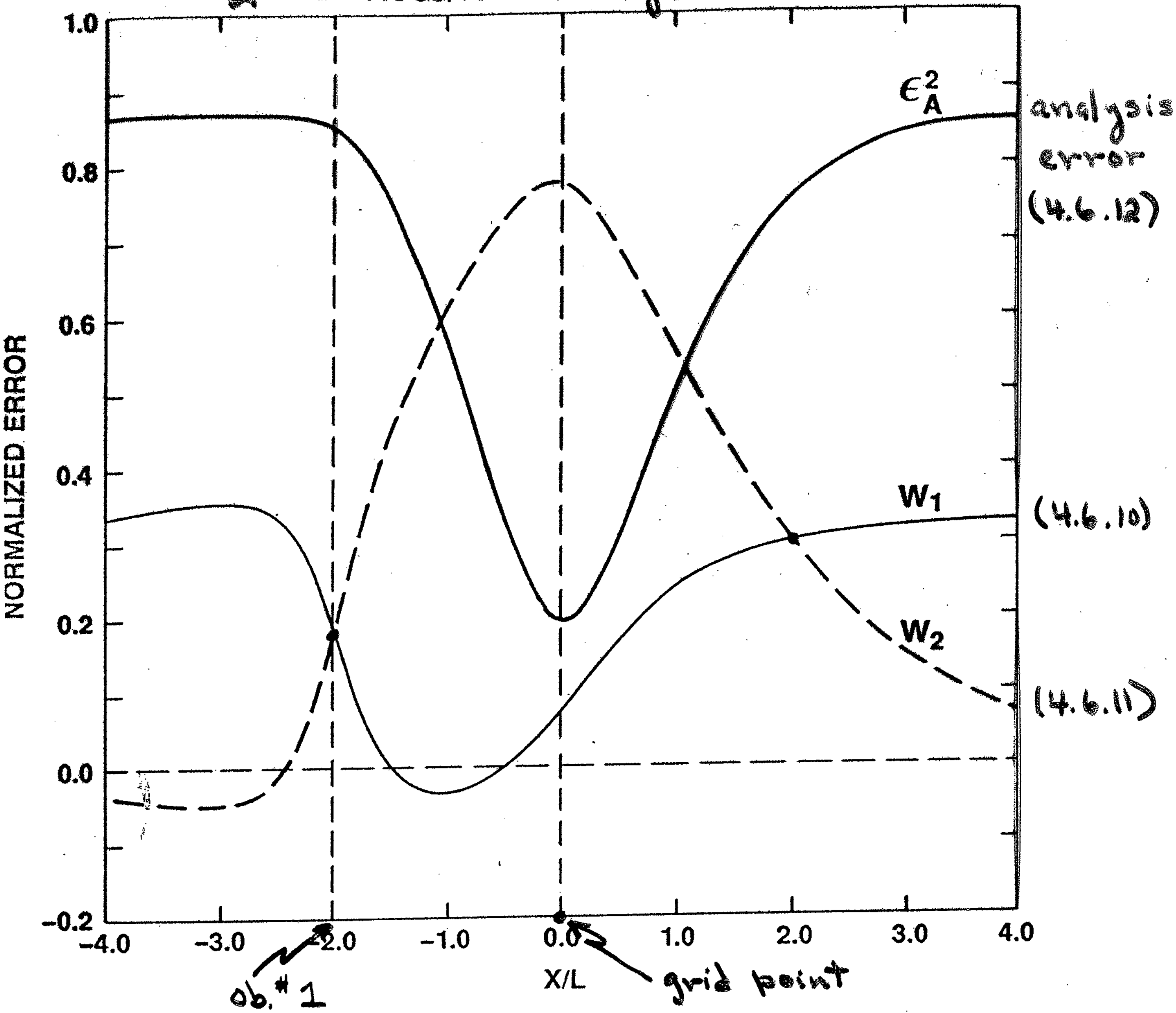
interpret

SCM would keep $W_{1,2,3}$ same for all positions on circle

$\therefore$ see how relative positions of obs. affects analysis

Go over Fig. 4.7 in Daley

2 Station Analysis




7 A posteriori weights $W_1$ and $W_2$ and normalized expected analysis error variance $\varepsilon_A^2$ for the analysis gridpoint at $x = 0$, observation 1 at $x = -2.0$ and the position of observation 2 varying between $x = \pm\infty$.

$$\varepsilon_1^2 = \varepsilon_2^2 = .25$$

(normalized observation error variance)

$$\rho_B(x) = \left(1 + \frac{|x|}{L}\right) \exp\left(-\frac{|x|}{L}\right)$$