Statistical Objective Analysis

Reading: Daley, p. 101-106

We shall follow Sec. 4.2 in Daley for the derivation of the Statistical Objective Analysis (SOA) technique, which is obtained via the minimum variance approach. In Sec. 4.1, Daley returns to the one-dimensional approach used in Sec. 2.2 (our Lectures #3 & 4) to show that the minimum variance estimate [2nd eq. (6), Lecture 3] has an expected analysis error variance that is less than the smallest expected observational error variance. Thus each observation, whatever its error, always reduces the expected error variance of the analysis. This indicates that all observations have some udefullness.

Derivation of SOA Algorithm

We start with the basic form of the analysis equation used in the SCM lectures:

$$f_A(\mathbf{r}_i) = f_B(\mathbf{r}_i) + \Sigma W_{ik}[f_o(\mathbf{r}_k) - f_B(\mathbf{r}_k)] \quad (1)$$

where $f_A(\mathbf{r}_i)$ is the analysis value at grid point i,

$f_B(\mathbf{r}_i)$ is the background field value at grid point i,

$f_o(\mathbf{r}_k)$ is the observation at the station,

$f_B(\mathbf{r}_k)$ is the background value at the station and

W_{ik} is the as yet undetermined *a posteriori* weight.

It is assumed that the points $\mathbf{r}_k$ belong to locations of irregularly-spaced observations, and points $\mathbf{r}_i$ are locations of regularly-spaced grid points.

We now introduce simpler notation: Define

$$A_i = f_A(\mathbf{r}_i)$$
$$B_i = f_B(\mathbf{r}_i)$$
$$O_k = f_o(\mathbf{r}_k)$$
$$B_k = f_B(\mathbf{r}_k)$$

Note that all of these quantities have error. Thus eq. (1) becomes

$$A_i = B_i + \sum_{k=1}^{K} W_{ik}[O_k - B_k] \quad (2)$$

Recall $A_i - B_i$ is the analysis increment, and $O_k - B_k$ is the observation increment.

Now define T_i as the true value of f at the grid points, and

T_k as the true value of f at the stations. Neither T_i or T_k are known.

[Note that T_i , T_k consist only of truth that can be resolved by the observing network; i.e., the "smooth truth"; see discussion in text]

Now subtract T_i from both sides of eq. (2):

$$A_i - T_i = B_i - T_i + \sum_{k=1}^{K} W_{ik} [O_k - B_k] \qquad (3)$$

Assume background and observational errors are unbiased:

$$< B_i - T_i > = < B_k - T_k > = < O_k - T_k > = 0 \qquad (4)$$

Thus the observation increment $< O_k - B_k >$ will be unbiased, and, via (3), $< A_i - T_i > = 0$ as well (analysis is unbiased).

Note that $< T >$ is the "true climate", and, in general, since the background usually comes from a forecast model, which may have bias, and since observations may have bias, then

$$< B > \neq < T > \text{ and } < O > \neq < T >.$$

However, since the bias is usually known, and we have learned how to remove bias (Lecture #4), we will assume that B_i , B_k and O_k are unbiased.

Now we can proceed with the derivation. The goal is to minimize the **expected analysis error variance**

$$E_A^2 = <(A_i - T_i)^2>,$$ which we need to obtain. Thus, we

square both sides of eq. (3):

$$(A_i - T_i)^2 = [(B_i - T_i) + \sum_{k=1}^{K} W_{ik} (O_k - B_k)]^2$$

$$= (B_i - T_i)^2 + 2\sum_{k=1}^{K} W_{ik} (O_k - B_k) (B_i - T_i) + [\sum_{k=1}^{K} W_{ik} (O_k - B_k)]^2$$

We now take expected values of this:

$$< (A_i - T_i)^2 > \;=\; < (B_i - T_i)^2 > \;+\; 2\sum_{k=1}^{K} W_{ik} < (O_k - B_k) (B_i - T_i) >$$

$$+ \sum_{k=1}^{K}\sum_{l=1}^{K} W_{ik}\, W_{il} < (O_k - B_k) (O_l - B_l) > \qquad (5)$$

We define $E_A^2 = < (A_i - T_i)^2 >$ as the <u>expected analysis error variance</u>

and $E_B^2 = < (B_i - T_i)^2 >$ as the <u>expected background error variance</u>

Recall that the <u>covariance</u> between two random variables s, q is

$$\text{Cov}\,(s, q) = < [s - <s>] [q - <q>] >$$

And that $\text{Cov}\,(s, s) = \text{Var}\, s = \sigma_s^2$.

Then $\text{Cov}\,[(O_k - B_k), (O_l - B_l)] =$

$$< [(O_k - B_k) - <(O_k - B_k)>] [(O_l - B_l) - <(O_l - B_l)>] > .$$

But since $< (O_k - B_k) > = <(O_l - B_l)> = 0$ (unbiased),

then $< (O_k - B_k) (O_l - B_l) >$ is the covariance between observation increments at station points $\mathbf{r}_k$ and $\mathbf{r}_l$.

Similarly, $< (O_k - B_k) (B_i - T_i) >$ is the covariance between the background error at i and the observation increment at k.

So, to minimize E_A^2, that is, to determine the weights W_{ik} that minimize eq. (5), we differentiate (5) with respect to each of the weights W_{ik} :

$$\frac{\partial E_A^2}{\partial W_{ik}} = + \; 2 < (O_k - B_k) (B_i - T_i) >$$

$$+ \; 2\sum_{l=1}^{K} W_{il} < (O_k - B_k) (O_l - B_l) > \;\Rightarrow\; 0 \qquad (6)$$

or

$$\sum_{l=1}^{K} W_{il} < (O_k - B_k) (O_l - B_l) > \;=\; - < (O_k - B_k) (B_i - T_i) > \qquad (7)$$

Consider the RHS of eq. (7), which can be rewritten **(class exercise)**

$$<(O_k - B_k)(B_i - T_i)> = <(O_k - T_k)(B_i - T_i)> - <(B_k - T_k)(B_i - T_i)> \tag{8}$$

Terms of the form $<(O_m - T_m)(B_n - T_n)>$ (i.e. – the covariance between observational error and the background error) might be expected to be zero (except when background fields are used to obtain observed fields such as in satellite retrievals). Here we will assume they are zero, and so the 2nd term on the RHS of (8) replaces the RHS of (7):

$$\sum_{l=1}^{K} W_{il} <(O_k - B_k)(O_l - B_l)> = <(B_k - T_k)(B_i - T_i)> \tag{9}$$

Now consider the covariance term on the LHS of (9). We first add two terms assumed to be zero:

$$<(O_k - B_k)(O_l - B_l)> + <(O_k - T_k)(B_l - T_l)> + <(B_k - T_k)(O_l - T_l)>$$

Since the expectation operator is commutative, we can put all terms inside one $<>$ operator, carry out all products, and then regroup the remaining terms to obtain

$$= <(B_k - T_k)(B_l - T_l)> + <(O_k - T_k)(O_l - T_l)> \tag{10}$$

Note that the first term on the RHS of (10) is the background error covariance term and the 2nd term is the observational error covariance term. Using this result in eq. (9), we have

$$\sum_{l=1}^{K} W_{il} [<(B_k - T_k)(B_l - T_l)> + <(O_k - T_k)(O_l - T_l)>] =$$

$$<(B_k - T_k)(B_i - T_i)> \tag{11}$$

Therefore, the statistical objective analysis method is the use of eq. (2) with the weights determined by eq. (11).

We now put eq. (11) in matrix form, using the following symbols:

$\underline{f}_o$ - a column vector of length K of the observations $f_o(\mathbf{r}_k)$

$\underline{f}_B$ - a column vector of length K of background values $f_B(\mathbf{r}_k)$ at stations

$\underline{\mathbf{W}}_i$ - a column vector of length K of *a posterori* weights (one for each grid point)

$\underline{\mathbf{B}}_i$ - a column vector of length K of the covariance between background error at stations and at grid points

$\underline{\underline{\mathbf{B}}}$ - background error covariance matrix with elements $< \varepsilon_B(\mathbf{r}_k)\, \varepsilon_B(\mathbf{r}_l) >$

$\underline{\underline{\mathbf{O}}}$ - observational error covariance matrix with elements $< \varepsilon_o(\mathbf{r}_k)\, \varepsilon_o(\mathbf{r}_l) >$

Both $\underline{\underline{\mathbf{B}}}$ and $\underline{\underline{\mathbf{O}}}$ are symmetric KxK matrices. Thus eq. (2) becomes

$$f_A(\mathbf{r}_i) = f_B(\mathbf{r}_i) + \underline{W}_i^T [\, \underline{f}_o - \underline{f}_B \,] \qquad (12)$$

and eq. (11) is

$$[\, \underline{\underline{\mathbf{B}}} + \underline{\underline{\mathbf{O}}} \,]\, \underline{W}_i = \underline{\mathbf{B}}_i \qquad (13)$$

Thus $\underline{W}_i$ are the optimal weights and the use of (13) into (10) gives us the optimal analysis equation:

$$f_A(\mathbf{r}_i) = f_B(\mathbf{r}_i) + \underline{\mathbf{B}}_i^T [\, \underline{\underline{\mathbf{B}}} + \underline{\underline{\mathbf{O}}} \,]^{-1} [\, \underline{f}_o - \underline{f}_B \,] \qquad (14)$$

which is the same as the optimal analysis equation we obtained at the end of Lecture #12.