MESOSCALE METEOROLOGY

METR 4433 | Problem Set #2 | Spring 2015

Assigned: Mar. 5, Due: Mar 26

1.) Dynamics (25 points)

The vertical equation of motion in an inviscid, Boussinesq atmosphere is given by

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \tag{1}$$

Let the dependent thermodynamic variables be given by

$$p(t,x,y,z) = \overline{p}(z) + p^{'}(t,x,y,z)$$
$$\rho(t,x,y,z) = \rho_0 + \rho^{'}(t,x,y,z),$$

where the first terms on the right-hand side denote the basic state, which is in hydrostatic balance, with a subscript "o" indicating a constant value.

Show that the Eq. (1) can be written

$$\frac{dw}{dt} = -\frac{1}{\rho_0}\frac{\partial p^{'}}{\partial z} + g\left(\frac{\theta^{'}}{\overline{\theta}} + 0.61q_v^{'}\right), \tag{2}$$

where q_v is the water vapor mixing ratio and the prime indicates a pressure perturbation from a basic state which is a function only of height (hint: linearize the pressure gradient force as we have previously done and use the virtual temperature to account for moisture effects).

Solution

$$\begin{aligned}
\frac{dw}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \\
&= -\frac{1}{(\rho_0 + \rho^{'})}\frac{\partial(\overline{p} + p^{'})}{\partial z} - g \\
&= -\frac{1}{\rho_0(1 + \rho^{'}/\rho_0)}\frac{\partial(\overline{p} + p^{'})}{\partial z} - g \\
&= -\frac{1}{\rho_0}\left(1 + \frac{\rho^{'}}{\rho_0}\right)^{-1}\frac{\partial(\overline{p} + p^{'})}{\partial z} - g
\end{aligned}$$

Since $\rho^{'}/\rho_0 \ll 1$, we can use the binomial approximation, which states that $(a+b)^{-1} \approx (a-b)$ if $b \ll 0$.

$$
\begin{aligned}
&= -\frac{1}{\rho_0}\left(1 - \frac{\rho^{'}}{\rho_0}\right)\frac{\partial(\overline{p} + p^{'})}{\partial z} - g \\
&= \underbrace{-\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial z}}_{g} - \frac{1}{\rho_0}\frac{p^{'}}{\partial z} + \frac{\rho^{'}}{\rho_0^2}\underbrace{\frac{\partial \overline{p}}{\partial z}}_{-\rho_0 g} + \underbrace{\frac{\rho^{'}}{\rho_0^2}\frac{\partial p^{'}}{\partial z}}_{\text{small, neglect}} - g \\
&= \cancel{g} - \frac{1}{\rho_0}\frac{p^{'}}{\partial z} - \frac{\rho^{'}}{\rho_0}g - \cancel{g} \\
&= -\frac{1}{\rho_0}\frac{p^{'}}{\partial z} - \frac{\rho^{'}}{\rho_0}g
\end{aligned}
$$

There are several ways to show that $-\rho^{'}/\rho_0 \approx \theta^{'}/\overline{\theta}$ (we showed this in a previous homework). Remember in the case of temperature and pressure and density that we are dealing with perturbations from a base state that satisfies the hydrostatic equation. Thus, $\overline{\theta}$ represents the base-state (or reference) temperature.

Since we are interested in moisture effects, we can replace potential temperature with virtual potential temperature, such that $-\rho^{'}/\rho_0 \approx \theta_v^{'}/\overline{\theta_v}$. Recall that

$$
\theta_v = \theta + 0.61\theta q = \theta + 0.61(\overline{\theta} + \theta^{'})q_v = \theta + 0.61\overline{\theta}\left(1 + \frac{\theta^{'}}{\overline{\theta}}\right)q_v \approx \theta + 0.61\overline{\theta}q_v
$$

Substitution yields

$$
\frac{\theta_v^{'}}{\overline{\theta_v}} = \frac{\theta^{'} + 0.61\overline{\theta}q^{'}}{\overline{\theta}} = \frac{\theta^{'}}{\overline{\theta}} + 0.61q_v^{'}
$$

Accordingly,

$$
-\frac{\rho^{'}}{\rho_0} \approx \frac{\theta^{'}}{\overline{\theta}} + 0.61q_v^{'}
$$

Thus, we arrive at

$$
\frac{dw}{dt} = -\frac{1}{\rho_0}\frac{p^{'}}{\partial z} + g\left(\frac{\theta^{'}}{\overline{\theta}} + 0.61q_v^{'}\right)
$$

2.) Thermodynamics – Adverse Pressure Gradients in Cloud Updrafts (25 points)

Consider a hypothetical air parcel that is moving vertically in an atmosphere with a constant potential temperature of 300 K. If the parcel is 4 K warmer than the environment at the ground and starts from a state of rest,

(a) Using the appropriate perturbation equation of motion (neglect friction, pressure, and moisture effects), find the vertical velocity of the parcel at an altitude of 4 km.

Solution
From the notes,

$$\frac{dw}{dt} = B \rightarrow d\left(\frac{w^2}{2}\right) = Bdz.$$

We integrate this from the surface to 4 km. Note that $B = g(\theta'/\overline{\theta})$ since moisture effects are neglected.

$$\int_0^{4\text{ km}} \frac{d}{dz}\left(\frac{w^2}{2}\right) dz = \int_0^{4\text{ km}} g\left(\frac{\theta'}{\overline{\theta}}\right) dz$$

From the problem statement, potential temperature is constant with height, thus $g(\theta'/\overline{\theta})$ is constant.

$$\frac{w_{4\text{ km}}^2 - \cancelto{0}{w_0^2}}{2} = g\left(\frac{\theta'}{\overline{\theta}}\right)(z_{4\text{ km}} - \cancelto{0}{z_0}) \quad \rightarrow \quad w_{4\text{ km}} = \sqrt{2g\left(\frac{\theta'}{\overline{\theta}}\right) z_{4\text{ km}}}$$

Plugging in the stated values yields

$$w_{4\text{ km}} = \sqrt{2(9.81\text{ ms}^{-2})\left(\frac{4\text{ K}}{300\text{ K}}\right)(4000\text{ m})} = \pm 32.3\text{ ms}^{-1}.$$

Since we are interested in upward velocity, the answer is $w_{4\text{ km}} = 32.3\text{ ms}^{-1}$.

(b) What is the magnitude and direction of the vertical perturbation pressure gradient force per unit mass (units of ms^{-2}) if the parcel attains an upward vertical velocity of 30 ms^{-1} at an altitude of 4 km?

solution
We now include pressure, $PGF = -(1/\rho_0)(\partial p'/\partial z)$, in our vertical equation of motion

$$\frac{dw}{dt} = PGF + B \rightarrow d\left(\frac{w^2}{2}\right) = PGF\,dz + Bdz.$$

Next, we integrate from the surface to 4 km, again recalling that B and PGF are constant with height

$$\frac{w_{4\text{ km}}^2 - \cancelto{0}{w_0^2}}{2} = \left[PGF + g\left(\frac{\theta'}{\overline{\theta}}\right)\right](z_{4\text{ km}} - \cancelto{0}{z_0})$$

Solving for PGF and plugging in known values yields

$$PGF = \frac{w_{4\text{ km}}^2}{2z_{4\text{ km}}} - g\left(\frac{\theta'}{\overline{\theta}}\right) = \frac{(30\text{ ms}^{-1})^2}{2(4000\text{ m})} - (9.81\text{ ms}^{-2})\left(\frac{4\text{ K}}{300\text{ K}}\right) = -0.018\text{ ms}^{-2}$$

This is a downward directed PGF. Sensible because the parcel slowed compared to part (a) answer.

(c) How long did it take for the parcel to reach an altitude of 4 km (hint: use $F = ma$ and recall that our two forces per unit mass are buoyancy and pressure gradient)?

Solution

$$F = ma \rightarrow \frac{F}{m} = a = \frac{V}{t}$$

Our two forces per unit mass are $B = (9.8 \text{ ms}^{-2})(4 \text{ K})/(300 \text{ K}) = 0.131 \text{ ms}^{-2}$ and $PGF = -0.018 \text{ ms}^{-2}$. Substitution and solving for t yields

$$t = \frac{30 \text{ ms}^{-1}}{0.131 \text{ ms}^{-2} - 0.018 \text{ ms}^{-2}} \sim 265 \text{ s} = 4.4 \text{ min}$$

3.) Density Currents (25 points)

Assume that the cold outflow from a squall line is propagating at a speed of 15 ms^{-1} into a calm environment. If the temperature within the outflow is 10 K cooler than the environment, with the environment having a temperature of 305K,

(a) Estimate the mean depth h of the density current (hint: start with Eq. (2) and neglect moisture effects).

Solution

For our density current, flow nearing the gust front can be considered horizontal. Thus, $dw/dt = 0$. Our equation of motion (neglecting moisture) becomes

$$\frac{1}{\rho_0}\frac{\partial p'}{\partial z} = g\frac{\theta'}{\overline{\theta}}$$

Now we integrate from the surface to the top of the density current h

$$\int_0^h \frac{1}{\rho_0}\frac{\partial p'}{\partial z}dz = \int_0^h g\frac{\theta'}{\overline{\theta}}dz$$
$$-\frac{\Delta p}{\rho_0} = g\frac{\theta'}{\overline{\theta}}h$$

Solving for h yields

$$h = -\frac{\overline{\theta}\Delta p}{g\rho_0\theta'}$$

What's Δp? Recall from class that $U = \sqrt{2\Delta p/\rho_0}$, thus $\Delta p = (\rho_0 U^2)/2$. Substitution yields

$$h = -\frac{\overline{\theta}\cancel{\rho_0}U^2}{2g\cancel{\rho_0}\theta'} = -\frac{\overline{\theta}U^2}{2g\theta'}$$

Plugging in known values give us

$$h = -\frac{(305 \text{ K})(15 \text{ ms}^{-1})^2}{2(9.8 \text{ ms}^{-2})(-10 \text{ K})} \sim 350 \text{ m}$$

(b) Estimate the pressure rise behind the gust front (in millibars) if the environmental density ρ_0 at the ground is 1.1 kg m^{-3}.

Solution

Again,

$$U = \sqrt{\frac{2\Delta p}{\rho_0}} \rightarrow \Delta p = \frac{\rho_0 U^2}{2}$$

Plugging in known values gives us

$$\Delta p = \frac{(15 \text{ ms}^{-1})^2 (1.1 \text{ kg m}^{-3})}{2} = 123.75 \text{ Pa} = 1.24 \text{ mb}$$

4.) Buoyancy (25 points)

For the parcel in Problem 2, determine the liquid water mixing ratio (in g/kg) that would be required to make this parcel neutrally buoyant (hint: consider water loading, ignore vapor buoyancy).

Solution

Recall from class that the combined buoyancy term (neglecting vapor contributions) is given by

$$B = g\left(\frac{\theta'}{\overline{\theta}} - l\right),$$

where l is the liquid water mixing ratio (g/kg). For neutral buoyancy, $B = 0$, and thus

$$\frac{\theta'}{\overline{\theta}} = l.$$

From Problem 2, $\theta'/\overline{\theta} = (4 \text{ K})/(300 \text{ K}) = 0.0133$. We need units of g/kg, so

$$0.013 \left[\frac{\text{kg}}{\text{kg}}\right] = 0.133 \left[\frac{\text{kg}}{\text{kg}}\right]\left[\frac{1000 \text{ g}}{\text{kg}}\right] = 13.3 \left[\frac{\text{g}}{\text{kg}}\right]$$

Thus, $l = 13.3 \text{ g}/\text{kg}$.