

MESOSCALE METEOROLOGY

METR 4433

Problem Set #1

Spring 2015

Answers

1.) Scale Analysis (10 points)

Using the appropriate horizontal and vertical momentum equations, perform a scale analysis for a mesoscale mountain disturbance and describe the main characteristics of motion at this scale (*i.e.*, what does each term tell you about the flow?). Hint: Assume Orlanski's definitions of mesoscale.

Solution

According to Orlanski's scale classification that we covered in Section 1.1.5 of the notes, mesoscale mountain disturbances are characterized as meso- β phenomena. This means we simply follow the procedure outlined in Section 1.1.8 of the notes.

The relevant appropriate scales are given by

- $V \sim 10 \text{ ms}^{-1}$
- $W \sim 1 \text{ ms}^{-1}$
- $L \sim 100 \text{ km} = 10^5 \text{ m}$
- $H \sim 10 \text{ km} = 10^4 \text{ m}$
- $T \sim L/V = 10^4 \text{ s}$
- $f \sim 10^{-4} \text{ s}^{-1}$
- $\rho \sim 1 \text{ kg m}^{-3}$
- Δp in horizontal $\sim 1 \text{ mb} = 100 \text{ Pa}$
- Δp in vertical $\sim 1000 \text{ mb} = 10^5 \text{ Pa}$

First we apply these scales to the horizontal momentum equation

$$\begin{array}{cccccc}
 \frac{\partial u}{\partial t} & +u \frac{\partial u}{\partial x} & +w \frac{\partial u}{\partial z} & = -\frac{1}{\rho} \frac{\partial p}{\partial x} & +fv \\
 \\
 \frac{V}{T} & \frac{VV}{L} & \frac{WV}{H} & \frac{\Delta p}{\rho L} & fV \\
 \\
 \frac{10}{10^4} & \frac{10^2}{10^5} & \frac{1 \times 10}{10^4} & \frac{10^2}{1 \times 10^5} & 10^{-4} \times 10 \\
 \\
 10^{-3} & 10^{-3} & 10^{-3} & 10^{-3} & 10^{-3}
 \end{array}$$

Next, we apply these scales to the vertical momentum equation

$$\begin{array}{rcccccc}
 \frac{\partial w}{\partial t} & +u \frac{\partial w}{\partial x} & +w \frac{\partial w}{\partial z} & = & -\frac{1}{\rho} \frac{\partial p}{\partial z} & -g \\
 \\
 \frac{W}{T} & \frac{VW}{L} & \frac{WW}{H} & & \frac{\Delta p}{\rho H} & g \\
 \\
 \frac{1}{10^4} & \frac{10 \times 1}{10^5} & \frac{1 \times 1}{10^4} & & \frac{10^5}{1 \times 10^4} & 10 \\
 \\
 10^{-4} & 10^{-4} & 10^{-4} & & 10 & 10
 \end{array}$$

Considering the horizontal momentum equation, we see that the all terms in the equation are of the same magnitude – none of them can be neglected. That means the horizontal pressure gradient force is no longer balanced by Coriolis. Thus, the flow is no longer geostrophic.

For the vertical momentum equation, the vertical pressure gradient force and gravitational acceleration are the dominant terms. That means the hydrostatic balance is still appropriate.

Based on the scales, the motion is quasi-two-dimensional ($w \ll u$ or v).

Thus, a mesoscale mountain disturbance is summarized as

- quasi-two-dimensional
- nearly hydrostatic
- Coriolis force is non-negligible

2.) Linear Perturbation Theory (10 points)

In Section 2.1.2, we developed a set of two-dimensional, irrotational, inviscid, and adiabatic equations of motion to describe internal atmospheric wave motions. In order to linearize the equations, we assumed the following base state:

$$\begin{aligned}
 u(t, x, z) &= \bar{u}(z) + u'(t, x, z) \\
 w(t, x, z) &= w'(t, x, z) \\
 p(t, x, z) &= \bar{p}(z) + p'(t, x, z) \\
 \theta(t, x, z) &= \bar{\theta}(z) + \theta'(t, x, z) \\
 \rho(t, x, z) &= \bar{\rho}(z) + \rho'(t, x, z) \quad \text{if paired with } g \\
 \rho(t, x, z) &= \bar{\rho}(z) \quad \text{otherwise}
 \end{aligned}$$

Use these base-state expressions to derive the linearized equations of motion given in the notes. That is, show:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial u'}{\partial t} + \bar{u} \frac{\partial u'}{\partial x} + w' \frac{\partial \bar{u}}{\partial z} + \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x} = 0 \quad (1)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial z} + g = \frac{\partial w'}{\partial t} + \bar{u} \frac{\partial w'}{\partial x} + \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z} - g \frac{\theta'}{\bar{\theta}} = 0 \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \quad (3)$$

$$\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + w \frac{\partial \theta}{\partial z} = \frac{\partial \theta'}{\partial t} + \bar{u} \frac{\partial \theta'}{\partial x} + w' \frac{N^2 \bar{\theta}}{g} = 0, \quad (4)$$

where N^2 is the Brunt-Väisälä frequency.

Show your work and make sure to describe the steps along the way. For instance, the gravity term in Eq.(2) was simply given in the notes. You will need to actually prove it. Hints: See section 2.3.3 in Markowski or section 7.4.1 in Holton (show your work - don't just take their word for it). Remember to ignore non-linear terms.

Solution

$$\begin{aligned} & \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \\ \rightarrow & \frac{\partial(\bar{u} + u')}{\partial t} + (\bar{u} + u') \frac{\partial(\bar{u} + u')}{\partial x} + w' \frac{\partial(\bar{u} + u')}{\partial z} + \frac{1}{\bar{\rho}} \frac{\partial(\bar{p} + p')}{\partial x} = 0 \\ \rightarrow & \underbrace{\frac{\partial \bar{u}}{\partial t}}_{(I)} + \frac{\partial u'}{\partial t} + \underbrace{\bar{u} \frac{\partial \bar{u}}{\partial x}}_{(I)} + \bar{u} \frac{\partial u'}{\partial x} + \underbrace{u' \frac{\partial \bar{u}}{\partial x}}_{(I)} + \underbrace{u' \frac{\partial u'}{\partial x}}_{(II)} + w' \frac{\partial \bar{u}}{\partial z} + \underbrace{w' \frac{\partial u'}{\partial z}}_{(II)} + \underbrace{\frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial x}}_{(I)} + \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x} = 0 \\ \rightarrow & \boxed{\frac{\partial u'}{\partial t} + \bar{u} \frac{\partial u'}{\partial x} + w' \frac{\partial \bar{u}}{\partial z} + \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x} = 0} \end{aligned}$$

The (I) terms are eliminated since $\bar{u} = \bar{u}(z)$ and thus, the time or horizontal derivatives are zero. Meanwhile, the (II) terms are eliminated since we were told to ignore non-linear terms.

$$\begin{aligned} & \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial z} + g = 0 \\ \rightarrow & \frac{\partial w'}{\partial t} + (\bar{u} + u') \frac{\partial w'}{\partial x} + w' \frac{\partial w'}{\partial z} + \frac{1}{\bar{\rho}} \frac{\partial(\bar{p} + p')}{\partial z} + g = 0 \\ \rightarrow & \frac{\partial w'}{\partial t} + \bar{u} \frac{\partial w'}{\partial x} + \underbrace{u' \frac{\partial w'}{\partial x}}_{(II)} + \underbrace{w' \frac{\partial w'}{\partial z}}_{(II)} + \frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial z} + \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z} + g = 0 \end{aligned}$$

The (II) terms are eliminated since they are non-linear. Let's look at the last three terms.

$$\frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial z} + \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z} + g$$

$$\rightarrow \frac{1}{(\bar{\rho} + \rho')} \left(\frac{(\bar{\rho} + \rho')}{\bar{\rho}} \frac{\partial \bar{p}}{\partial z} + \frac{(\bar{\rho} + \rho')}{\bar{\rho}} \frac{\partial p'}{\partial z} + \bar{\rho}g + \rho'g \right)$$

Since $(\bar{\rho} + \rho')/\bar{\rho} \approx 1$, we have

$$\frac{1}{(\bar{\rho} + \rho')} \left(\frac{\partial \bar{p}}{\partial z} + \frac{\partial p'}{\partial z} + \bar{\rho}g + \rho'g \right)$$

Recall that the base state is hydrostatic, so $\partial \bar{p}/\partial z = -\bar{\rho}g$, which leaves

$$\frac{1}{(\bar{\rho} + \rho')} \left(\frac{\partial p'}{\partial z} + \rho'g \right) = \frac{1}{\bar{\rho} \left(1 + \frac{\rho'}{\bar{\rho}} \right)} \left(\frac{\partial p'}{\partial z} + \rho'g \right)$$

Again, $1 + \rho'/\bar{\rho} \approx 1$, which yields

$$\frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial z} + \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z} + g = \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z} + \frac{\rho'}{\bar{\rho}}g$$

I will not reproduce the derivation here, but you can follow Section 3.3.1 in the notes (or the other suggested sources) and show that

$$\frac{\rho'}{\bar{\rho}} \approx -\frac{\theta'}{\bar{\theta}}$$

Putting it all together leads to

$$\boxed{\frac{\partial w'}{\partial t} + \bar{u} \frac{\partial w'}{\partial x} + \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z} - g \frac{\theta'}{\bar{\theta}} = 0}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \rightarrow \frac{\partial \bar{u}}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$\xrightarrow{(I)}$

$$\rightarrow \boxed{\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0}$$

The (I) term is eliminated since $\bar{u} = \bar{u}(z)$ and thus, the horizontal derivative is zero.

$$\begin{aligned}
& \frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + w \frac{\partial \theta}{\partial z} = 0 \\
& \rightarrow \frac{\partial(\bar{\theta} + \theta')}{\partial t} + (\bar{u} + u') \frac{\partial(\bar{\theta} + \theta')}{\partial x} + w' \frac{\partial(\bar{\theta} + \theta')}{\partial z} = 0 \\
& \rightarrow \underbrace{\frac{\partial \bar{\theta}}{\partial t}}_{(I)} + \frac{\partial \theta'}{\partial t} + \underbrace{\bar{u} \frac{\partial \bar{\theta}}{\partial x}}_{(I)} + \bar{u} \frac{\partial \theta'}{\partial x} + \underbrace{u' \frac{\partial \bar{\theta}}{\partial x}}_{(I)} + \underbrace{u' \frac{\partial \theta'}{\partial x}}_{(II)} + w' \frac{\partial \bar{\theta}}{\partial z} + \underbrace{w' \frac{\partial \theta'}{\partial z}}_{(II)} = 0 \\
& \rightarrow \frac{\partial \theta'}{\partial t} + \bar{u} \frac{\partial \theta'}{\partial x} + w' \frac{\partial \bar{\theta}}{\partial z} = 0 \\
& \rightarrow \boxed{\frac{\partial \theta'}{\partial t} + \bar{u} \frac{\partial \theta'}{\partial x} + w' \frac{N^2 \bar{\theta}}{g} = 0}
\end{aligned}$$

where $N^2 = (g/\bar{\theta})(\partial \bar{\theta}/\partial z)$. The (I) terms are eliminated since $\bar{u} = \bar{u}(z)$ and thus, the time or horizontal derivatives are zero. Meanwhile, the (II) terms are eliminated since we were told to ignore non-linear terms.

3.) Flows Over Two-Dimensional Sinusoidal Mountains (15 points)

- (a) Combine Eqs.(1)-(4) to derive Scorer's equation. Make sure to define the equation in terms of the Scorer parameter. Assume the flow is steady-state and that both the mean wind and static stability are constant with height.

Solution

Since the forcing for the waves is stationary, we further assume a steady state ($\partial/\partial t = 0$). Our equations reduce to

$$\bar{u} \frac{\partial u'}{\partial x} + w' \frac{\partial \bar{u}}{\partial z} + \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x} = 0 \quad (5)$$

$$\bar{u} \frac{\partial w'}{\partial x} + \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z} - g \frac{\theta'}{\bar{\theta}} = 0 \quad (6)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \quad (7)$$

$$\bar{u} \frac{\partial \theta'}{\partial x} + w' \frac{N^2 \bar{\theta}}{g} = 0, \quad (8)$$

Next, we take $\partial(\text{Eq. 5})/\partial z$ and $\partial(\text{Eq. 6})/\partial x$.

$$\bar{u} \frac{\partial}{\partial x} \frac{\partial u'}{\partial z} + \frac{1}{\bar{\rho}} \frac{\partial^2 p'}{\partial x \partial z} + w' \frac{\partial^2 \bar{u}}{\partial z^2} = 0 \quad (9)$$

$$\bar{u} \frac{\partial}{\partial x} \frac{\partial w'}{\partial x} + \frac{1}{\bar{\rho}} \frac{\partial^2 p'}{\partial x \partial z} - \frac{g}{\bar{\theta}} \frac{\partial \theta'}{\partial x} = 0. \quad (10)$$

Now, we subtract Eq. (9) from Eq. (10) in order to eliminate p' , which yields

$$\bar{u} \frac{\partial}{\partial x} \left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z} \right) - w' \frac{\partial^2 \bar{u}}{\partial z^2} - \frac{g}{\theta} \frac{\partial \theta'}{\partial x} . \quad (11)$$

Next, use Eq. (8) to eliminate $\partial \theta' / \partial x$:

$$\bar{u} \frac{\partial}{\partial x} \left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z} \right) + \left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z} \right) w' \frac{N^2}{\bar{u}} - \left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z} \right) w' \frac{\partial^2 \bar{u}}{\partial z^2} = 0 . \quad (12)$$

To further simplify the expression, we divide Eq. (12) by \bar{u} and distribute $\partial / \partial x$, which yields

$$\frac{\partial^2 w'}{\partial x^2} - \frac{\partial}{\partial z} \frac{\partial u'}{\partial x} + w' \left(\frac{N^2}{\bar{u}^2} - \frac{1}{\bar{u}} \frac{\partial^2 \bar{u}}{\partial z^2} \right) = 0 . \quad (13)$$

Finally, use Eq. (7) to eliminate $\partial u' / \partial x$ and arrive at *Scorer's equation* (1954):

$$\nabla^2 w' + l^2(z) w' = 0 . \quad (14)$$

Here, $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial z^2$ is the two-dimensional Laplacian operator and l is the **Scorer parameter** (Scorer 1949), which is defined by:

$$l(z) = \sqrt{\frac{N^2}{\bar{u}^2} - \frac{1}{\bar{u}} \frac{\partial^2 \bar{u}}{\partial z^2}} \quad (15)$$

(b) We found in class that the general solution to Scorer's equation for this flow is

$$w' = A \exp(i[kx + mz]) + B \exp(i[kx - mz]) ,$$

where $m = \sqrt{l^2 - k^2}$. Derive the two solutions for w' for the cases where $l > k$ and $l < k$, respectively. Describe the type of wave associated with each solution. Hint: The notes will be especially helpful.

Solution

For $l > k$: recall that the top boundary condition for this problem states that waves must transport energy away from their energy source (avoid reflection). It can be shown that solution waves of the form $\exp(i[kx + mz])$ have an energy flux that is directed upward, while those of the form $\exp(i[kx - mz])$ have a downward energy flux. In order to satisfy our upper boundary condition, this implies $B = 0$, leaving a solution of the form

$$w' = A \exp(i[kx + mz]) = A_1 \cos(kx + mz) + A_2 \sin(kx + mz) . \quad (16)$$

To solve for A_1 and A_2 , we invoke the lower boundary condition. Recall that the terrain behaves according to

$$h(x) = h_m \sin kx , \quad (17)$$

where h_m is mountain height. Then, because the flow at the lower boundary must be parallel to the boundary, the vertical velocity perturbation at the boundary is given by the rate at which the boundary height changes following the motion:

$$w'(x, 0) = \frac{Dh}{Dt}_{z=0} \approx \bar{u} \frac{\partial h}{\partial x} = \bar{u} k h_m \cos(kx) \quad (18)$$

Now we compare Eq. (18) and Eq. (16) at $z = 0$

$$w'(x, 0) = \bar{u} k h_m \cos(kx) = A_1 \cos(kx) + A_2 \sin(kx) . \quad (19)$$

This implies $A_1 = \bar{u} k h_m$ and $A_2 = 0$. Thus, the solution is

$$\boxed{w' = \bar{u} k h_m \cos(kx + mz) .} \quad (20)$$

This describes a wave that propagates vertically without loss of amplitude.

For $l < k$: we set $m = i\mu$, where $\mu = |m|$ is a real number. If we substitute this expression into the general solution to Scorer's equation, we arrive at our expected solution:

$$w' = A \exp(ikx) \exp(-\mu z) + B \exp(ikx) \exp(\mu z) . \quad (21)$$

We first apply our upper boundary condition to help determine the coefficients. In this case, the B term represents a wave whose amplitude grows exponentially unbounded with height. This is unphysical since the energy source is the mountain surface. This implies that $B = 0$, leaving a solution of the form

$$w' = A \exp(ikx) \exp(-\mu z) = A_1 \exp(-\mu z) \cos(kx) + A_2 \exp(-\mu z) \sin(kx) . \quad (22)$$

Now we compare Eq. (18) and Eq. (22) at $z = 0$

$$w'(x, 0) = \bar{u} k h_m \cos(kx) = A_1 \cos(kx) + A_2 \sin(kx) . \quad (23)$$

This again implies that $A_1 = \bar{u} k h_m$ and $A_2 = 0$. Thus, the solution is

$$\boxed{w' = \bar{u} k h_m \exp(-\mu z) \cos(kx) .} \quad (24)$$

This describes a wave that decays exponentially with height.

- (c) What type of atmospheric phenomenon occurs if we allow the mean wind and static stability to vary with height such that the Scorer parameter decreases rapidly with height?

Solution

Trapped lee waves

4.) Blocking of Wind by Terrain (15 points)

In class, we derived the combined Bernoulli equation for a streamline approaching terrain

$$\Pi' + \bar{\Pi} + \frac{u^2}{2c_p\theta_0} + \frac{gz}{c_p\theta_0} = \Pi_0 + \frac{u_0^2}{2c_p\theta_0} + \frac{gz_0}{c_p\theta_0},$$

where the upstream value of $\bar{\Pi}$ is given by

$$\bar{\Pi}(z) = \Pi_0 - \frac{g\delta}{c_p\theta_0} + \frac{N^2\delta^2}{2c_p\theta_0},$$

and δ is the vertical displacement of a streamline as it approaches the terrain.

- (a) Use these expressions to derive an equation that describes the speed of an air parcel (u^2) approaching the terrain. Assume that pressure perturbations are small enough to neglect.

Solution

If we plug $\bar{\Pi}(z)$ into the combined Bernoulli equation:

$$\cancel{\Pi'} + \cancel{\Pi_0} - \cancel{\frac{g\delta}{c_p\theta_0}} + \frac{N^2\delta^2}{2c_p\theta_0} + \frac{u^2}{2c_p\theta_0} + \cancel{\frac{gz}{c_p\theta_0}} = \cancel{\Pi_0} + \frac{u_0^2}{2c_p\theta_0} + \cancel{\frac{gz_0}{c_p\theta_0}}$$

The resulting expression is easily rearranged as

$$u^2 = u_0^2 - 2c_p\theta_0\Pi' - N^2\delta^2. \quad (25)$$

If $\Pi' = 0$ (ignore pressure perturbations), then

$$\boxed{u^2 = u_0^2 - N^2\delta^2}. \quad (26)$$

- (b) Using this equation, derive the maximum vertical displacement that is allowed before the flow stagnates.

Solution

The flow will stagnate when $u \rightarrow 0$.

$$u_0^2 - N^2\delta^2 = 0. \quad (27)$$

We now solve for δ (which corresponds to the maximum vertical displacement that is allowed before the flow stagnates)

$$\boxed{\delta_{max} = \frac{u_0}{N}}. \quad (28)$$

- (c) The mountain Froude number is defined as $F_{rm} = u/(Nh_m)$, where h_m is the mountain height. Redefine F_{rm} in terms of the maximum vertical displacement. Using this mountain Froude number, describe what you would expect in terms of blocking for flow over the mountain if $F_{rm} > 1$ and $F_{rm} < 1$. Be sure to relate the equation to a physical reason for each case.

Solution

$$F_{rm} = u/(Nh_m) \rightarrow \boxed{F_{rm} = \frac{\delta_{max}}{h_m}}$$

For $F_{rm} > 1$: this implies that the maximum vertical displacement that a parcel can traverse before stagnating is above than the mountain. This means that air will flow over the mountain.

For $F_{rm} < 1$: this implies that the maximum vertical displacement that a parcel can traverse before stagnating is below than the mountain. This means that air will be at least partially blocked by the mountain.

- (d) What is the main problem with using this approach to assess blocking potential?

Solution

The main weakness with using this formulation of F_{rm} to assess blocking potential is that is too simplistic to properly describe whether or not terrain will block the flow. Reasons include:

- Eq. (26) predicts that the minimum wind speeds exist at the crest of the mountain. Observations, however, show that the minimum wind speeds often occur halfway up the windward side of the mountain.
- Observations and simulations both indicate that terrain slope and aspect ratio (crosswise divided by streamwise) are critical factors affecting the tendency for blocking. For instance, blocking is more likely as the slope and aspect ratio of the terrain increase.

The reason for these discrepancies is that pressure perturbations are very important (if not the dominant factor) in determining an air parcel's acceleration as it approaches a barrier. Thus, *the main problem is that we neglect pressure perturbations.*