3.3.1 Reynolds Averaging, continued

Now we apply the Reynolds averaging rules to the momentum, thermodynamic, and moisture equations. Here the Boussinesq approximation is applied, such that the continuity equation is replaced by the incompressibility condition ($\nabla \cdot \vec{u} = 0$), density is assumed constant ($\overline{\rho}$) except when coupled with gravitational acceleration, and the fluid's viscosity, thermal diffusivity, and specific heat are assumed constant.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{1}{\overline{\rho}}\frac{\partial p}{\partial x} - fv - \nu\nabla^2 u = 0 \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{1}{\overline{\rho}}\frac{\partial p}{\partial y} + fu - \nu\nabla^2 v = 0 \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{1}{\overline{\rho}}\frac{\partial p}{\partial z} + g - \nu\nabla^2 w = 0 \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z} + \frac{1}{c_p\overline{\rho}}\nabla \cdot \vec{F_r} - \frac{1}{c_p\overline{\rho}}(H_{\text{so}} - H_{\text{si}}) - \nu_h\nabla^2\theta = 0 \tag{5}$$

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} + w\frac{\partial q}{\partial z} - \frac{1}{\overline{\rho}}(W_{\text{so}} - W_{\text{si}}) - \nu_q\nabla^2 q = 0\,. \tag{6}$$

In Eqs.(1) and (2), f is the Coriolis parameter. In Eqs.(1)-(3) ν is the kinematic viscosity. In Eq.(5), $\vec{F_r}$ is the radiative flux vector, ν_h is thermal diffusivity, and the subscripts "so" and "si" in the H terms stand, respectively, for sources and sinks of heat expressed in units of energy per unit volume per unit time. In Eq.(6), ν_q is moisture diffusivity and the subscripts "so" and "si" in the W terms stand, respectively, for sources and sinks of moisture expressed in units of mass per unit volume per unit time. Finally, c_p is the specific heat of air at constant pressure in Eqs.(5) and (6).

The radiative flux F_r depends on the properties of the atmosphere with respect to propagation of electromagnetic radiation. It is usually subdivided into the shortwave (solar/visible radiation) and longwave (atmospheric and terrestrial radiation) components. The divergence of radiative flux ($\nabla \cdot \vec{F_r}$) is often neglected in boundary layer theories and models (usually, without any sufficient supporting arguments). We will retain it here for completeness.

For each of these balance equations, we will apply Reynolds decomposition (*i.e.*, breaking variables into mean and fluctuating parts, *e.g.*, $\psi = \overline{\psi} + \psi'$). Reynolds averaging rules are then applied to the resulting expressions.

Zonal momentum balance equation

First, we will recast Eq.(1) into flux form by using the product rule for derivatives. Recall that

$$\frac{\partial (ab)}{\partial x} = a\frac{\partial b}{\partial x} + b\frac{\partial a}{\partial x}.$$

To achieve this form, multiply Eq.(4) by u and add to Eq.(1). This yields

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y} + w\frac{\partial u}{\partial z} + u\frac{\partial w}{\partial z} + \frac{1}{\overline{\rho}}\frac{\partial p}{\partial x} - fv - \nu\nabla^2 u = 0\ . \tag{7}$$

The product rule is applied, which leads the flux form of the zonal momentum balance equation

$$\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} + \frac{1}{\overline{\rho}}\frac{\partial p}{\partial x} - fv - \nu\nabla^2 u = 0\ . \tag{8}$$

Now, quantities are decomposed into mean and fluctuating parts

$$\begin{aligned}\frac{\partial \overline{u}}{\partial t} + \frac{\partial u^{'}}{\partial t} + \frac{\partial}{\partial x}&\left(\overline{u}\,\overline{u} + \overline{u}u^{'} + u^{'}\overline{u} + u^{'}u^{'}\right)\\ &+ \frac{\partial}{\partial y}\left(\overline{u}\,\overline{v} + \overline{u}v^{'} + u^{'}\overline{v} + u^{'}v^{'}\right)\\ &+ \frac{\partial}{\partial z}\left(\overline{u}\,\overline{w} + \overline{u}w^{'} + u^{'}\overline{w} + u^{'}w^{'}\right)\\ &+ \frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x} + \frac{1}{\overline{\rho}}\frac{\partial p^{'}}{\partial x} - f\overline{v} - fv^{'}\\ &- \nu\nabla^2\overline{u} - \nu\nabla^2 u^{'} = 0\ .\end{aligned} \tag{9}$$

Taking the Reynolds average of Eq.(9) yields

$$\begin{aligned}\overline{\frac{\partial \overline{u}}{\partial t}} + \overline{\frac{\partial u^{'}}{\partial t}} + &\overline{\frac{\partial}{\partial x}\left(\overline{u}\,\overline{u} + \overline{u}u^{'} + u^{'}\overline{u} + u^{'}u^{'}\right)}\\ &+ \overline{\frac{\partial}{\partial y}\left(\overline{u}\,\overline{v} + \overline{u}v^{'} + u^{'}\overline{v} + u^{'}v^{'}\right)}\\ &+ \overline{\frac{\partial}{\partial z}\left(\overline{u}\,\overline{w} + \overline{u}w^{'} + u^{'}\overline{w} + u^{'}w^{'}\right)}\\ &+ \overline{\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x}} + \overline{\frac{1}{\overline{\rho}}\frac{\partial p^{'}}{\partial x}} - \overline{f\overline{v}} - \overline{fv^{'}}\\ &- \overline{\nu\nabla^2\overline{u}} - \overline{\nu\nabla^2 u^{'}} = 0.\end{aligned} \tag{10}$$

We use the previously defined Reynolds averaging rules to simplify this equation. For sake of completeness, we will consider every term in Eq.(30) to full understand the process. For the remaining balance equations, this step is skipped and left to you as an exercise.

- $\overline{\dfrac{\partial \overline{u}}{\partial t}} = \dfrac{\partial \overline{\overline{u}}}{\partial t} = \dfrac{\partial \overline{u}}{\partial t}$
- $\overline{\dfrac{\partial u'}{\partial t}} = \dfrac{\partial \overline{u'}}{\partial t} = 0$
- $\overline{\dfrac{\partial \overline{u}\,\overline{u}}{\partial x}}\,,\ \overline{\dfrac{\partial \overline{u}\,\overline{v}}{\partial y}}\,,\ \overline{\dfrac{\partial \overline{u}\,\overline{w}}{\partial z}} = \dfrac{\partial \overline{\overline{u}\,\overline{u}}}{\partial x}\,,\ \dfrac{\partial \overline{\overline{u}\,\overline{v}}}{\partial y}\,,\ \dfrac{\partial \overline{\overline{u}\,\overline{w}}}{\partial z} = \dfrac{\partial \overline{u}\,\overline{u}}{\partial x}\,,\ \dfrac{\partial \overline{u}\,\overline{v}}{\partial y}\,,\ \dfrac{\partial \overline{u}\,\overline{w}}{\partial z}$
- $\overline{\dfrac{\partial \overline{u}u'}{\partial x}}\,,\ \overline{\dfrac{\partial \overline{u}v'}{\partial y}}\,,\ \overline{\dfrac{\partial \overline{u}w'}{\partial z}} = \dfrac{\partial \overline{\overline{u}u'}}{\partial x}\,,\ \dfrac{\partial \overline{\overline{u}v'}}{\partial y}\,,\ \dfrac{\partial \overline{\overline{u}w'}}{\partial z} = \dfrac{\partial \overline{u}\overline{u'}}{\partial x}\,,\ \dfrac{\partial \overline{u}\overline{v'}}{\partial y}\,,\ \dfrac{\partial \overline{u}\overline{w'}}{\partial z} = 0\ ,\ 0\ ,\ 0$
- $\overline{\dfrac{\partial u'\overline{u}}{\partial x}}\,,\ \overline{\dfrac{\partial u'\overline{v}}{\partial y}}\,,\ \overline{\dfrac{\partial u'\overline{w}}{\partial z}} = 0\ ,\ 0\ ,\ 0$ (same as above)
- $\overline{\dfrac{\partial u'u'}{\partial x}}\,,\ \overline{\dfrac{\partial u'v'}{\partial y}}\,,\ \overline{\dfrac{\partial u'w'}{\partial z}} = \dfrac{\partial \overline{u'u'}}{\partial x}\,,\ \dfrac{\partial \overline{u'v'}}{\partial y}\,,\ \dfrac{\partial \overline{u'w'}}{\partial z}$
- $\overline{\dfrac{1}{\overline{\rho}}\dfrac{\partial \overline{p}}{\partial x}} + \overline{\dfrac{1}{\overline{\rho}}\dfrac{\partial p'}{\partial x}} = \dfrac{1}{\overline{\rho}}\dfrac{\partial \overline{\overline{p}}}{\partial x} + \dfrac{1}{\overline{\rho}}\dfrac{\partial \overline{p'}}{\partial x} = \dfrac{1}{\overline{\rho}}\dfrac{\partial \overline{p}}{\partial x} + 0 = \dfrac{1}{\overline{\rho}}\dfrac{\partial \overline{p}}{\partial x}$
- $-\overline{f\overline{v}} - \overline{fv'} = -f\overline{\overline{v}} - f\overline{v'} = -f\overline{v} + 0 = -f\overline{v}$
- $-\overline{\nu\nabla^2\overline{u}} - \overline{\nu\nabla^2 u'} = -\nu\nabla^2\overline{\overline{u}} - \nu\nabla^2\overline{u'} = -\nu\nabla^2\overline{u} - 0 = -\nu\nabla^2\overline{u}$

Combining these results yields

$$\frac{\partial \overline{u}}{\partial t} + \frac{\partial \overline{u}\,\overline{u}}{\partial x} + \frac{\partial \overline{u}\,\overline{v}}{\partial y} + \frac{\partial \overline{u}\,\overline{w}}{\partial z} + \frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z} + \frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x} - f\overline{v} - \nu\nabla^2\overline{u} = 0\ . \tag{11}$$

Next, we decompose Eq.(4) and take the Reynolds average using the previously shown logic

$$\overline{\frac{\partial \overline{u}}{\partial x}} + \overline{\frac{\partial u'}{\partial x}} + \overline{\frac{\partial \overline{v}}{\partial y}} + \overline{\frac{\partial v'}{\partial y}} + \overline{\frac{\partial \overline{w}}{\partial z}} + \overline{\frac{\partial w'}{\partial z}} = \frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial z} + \frac{\partial \overline{w}}{\partial z} = 0\ . \tag{12}$$

Finally, we multiply Eq.(12) by $\overline{u}$ and subtract the result from Eq.(11) in order to revert from flux form

$$\frac{\partial \overline{u}}{\partial t} + \frac{\partial \overline{u}\,\overline{u}}{\partial x} - \overline{u}\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{u}\,\overline{v}}{\partial y} - \overline{u}\frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{u}\,\overline{w}}{\partial z} - \overline{u}\frac{\partial \overline{w}}{\partial z} + \frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z} + \frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x} - f\overline{v} - \nu\nabla^2\overline{u} = 0\ . \tag{13}$$

Using the product rule of derivatives, we arrive at the Reynolds-averaged zonal momentum equation

$$\boxed{\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} + \frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z} + \frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x} - f\overline{v} - \nu\nabla^2\overline{u} = 0} \tag{14}$$

Meridional momentum balance equation

Applying the same procedure as that used for the zonal momentum equation, the Reynolds-averaged form of Eq.(2) is given by

$$\boxed{\frac{\partial \overline{v}}{\partial t}+\overline{u}\frac{\partial \overline{v}}{\partial x}+\overline{v}\frac{\partial \overline{v}}{\partial y}+\overline{w}\frac{\partial \overline{v}}{\partial z}+\frac{\partial \overline{v^{'}u^{'}}}{\partial x}+\frac{\partial \overline{v^{'}v^{'}}}{\partial y}+\frac{\partial \overline{v^{'}w^{'}}}{\partial z}+\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial y}+f\overline{u}-\nu\nabla^2\overline{v}=0} \tag{15}$$

Vertical momentum balance equation

First, we apply Reynolds decomposition to Eq.(3) in the same way as was done for Eqs.(1) and (2)

$$\begin{aligned}\frac{\partial \overline{w}}{\partial t}+\frac{\partial w^{'}}{\partial t}+\frac{\partial}{\partial x}&\left(\overline{w}\,\overline{u}+\overline{w}u^{'}+w^{'}\overline{u}+w^{'}u^{'}\right)\\ &+\frac{\partial}{\partial y}\left(\overline{w}\,\overline{v}+\overline{w}v^{'}+w^{'}\overline{v}+w^{'}v^{'}\right)\\ &+\frac{\partial}{\partial z}\left(\overline{w}\,\overline{w}+\overline{w}w^{'}+w^{'}\overline{w}+w^{'}w^{'}\right)\\ &+\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial z}+\frac{1}{\overline{\rho}}\frac{\partial p^{'}}{\partial z}+g\\ &-\nu\nabla^2\overline{w}-\nu\nabla^2 w^{'}=0\,.\end{aligned} \tag{16}$$

As an aside, Eq.(16) is often written in terms of buoyancy. To do this, multiply Eq.(16) by $\rho=\overline{\rho}+\rho^{'}$

$$\begin{aligned}(\overline{\rho}+\rho^{'})\frac{\partial \overline{w}}{\partial t}+(\overline{\rho}+\rho^{'})\frac{\partial w^{'}}{\partial t}+(\overline{\rho}+\rho^{'})&\frac{\partial}{\partial x}\left(\overline{w}\,\overline{u}+\overline{w}u^{'}+w^{'}\overline{u}+w^{'}u^{'}\right)\\ &+(\overline{\rho}+\rho^{'})\frac{\partial}{\partial y}\left(\overline{w}\,\overline{v}+\overline{w}v^{'}+w^{'}\overline{v}+w^{'}v^{'}\right)\\ &+(\overline{\rho}+\rho^{'})\frac{\partial}{\partial z}\left(\overline{w}\,\overline{w}+\overline{w}w^{'}+w^{'}\overline{w}+w^{'}w^{'}\right)\\ &+\frac{(\overline{\rho}+\rho^{'})}{\overline{\rho}}\frac{\partial \overline{p}}{\partial z}+\frac{(\overline{\rho}+\rho^{'})}{\overline{\rho}}\frac{\partial p^{'}}{\partial z}+\overline{\rho}g+\rho^{'}g\\ &+(\overline{\rho}+\rho^{'})\left(-\nu\nabla^2\overline{w}-\nu\nabla^2 w^{'}\right)=0\,.\end{aligned} \tag{17}$$

For the pressure terms on the fourth line,

$$\frac{(\overline{\rho}+\rho^{'})}{\overline{\rho}}=\left(1+\frac{\rho^{'}}{\overline{\rho}}\right)\approx 1,$$

which leaves

$$\frac{\partial \overline{p}}{\partial z}+\frac{\partial p^{'}}{\partial z}+\overline{\rho}g+\rho^{'}g.$$

Recall that the base state atmosphere is hydrostatic, meaning

$$\frac{\partial \overline{p}}{\partial z} = -\overline{\rho} g,$$

which reduces the expression to

$$\frac{\partial p^{'}}{\partial z} + \rho^{'} g.$$

We apply these approximations and divide the resulting form of Eq.(17) by $\overline{\rho}$, which yields

$$\begin{aligned}
\frac{(\overline{\rho}+\rho^{'})}{\overline{\rho}}\frac{\partial \overline{w}}{\partial t} + \frac{(\overline{\rho}+\rho^{'})}{\overline{\rho}}\frac{\partial w^{'}}{\partial t} &+ \frac{(\overline{\rho}+\rho^{'})}{\overline{\rho}}\frac{\partial}{\partial x}\left(\overline{w}\,\overline{u} + \overline{w}u^{'} + w^{'}\overline{u} + w^{'}u^{'}\right) \\
&+ \frac{(\overline{\rho}+\rho^{'})}{\overline{\rho}}\frac{\partial}{\partial y}\left(\overline{w}\,\overline{v} + \overline{w}v^{'} + w^{'}\overline{v} + w^{'}v^{'}\right) \\
&+ \frac{(\overline{\rho}+\rho^{'})}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{w}\,\overline{w} + \overline{w}w^{'} + w^{'}\overline{w} + w^{'}w^{'}\right) \\
&+ \frac{1}{\overline{\rho}}\frac{\partial p^{'}}{\partial z} + \frac{\rho^{'}}{\overline{\rho}} g \\
&+ \frac{(\overline{\rho}+\rho^{'})}{\overline{\rho}}\left(-\nu\nabla^2\overline{w} - \nu\nabla^2 w^{'}\right) = 0\,.
\end{aligned} \tag{18}$$

We again make use of the fact that $(\overline{\rho}+\rho^{'})/\overline{\rho} \approx 1$,

$$\begin{aligned}
\frac{\partial \overline{w}}{\partial t} + \frac{\partial w^{'}}{\partial t} &+ \frac{\partial}{\partial x}\left(\overline{w}\,\overline{u} + \overline{w}u^{'} + w^{'}\overline{u} + w^{'}u^{'}\right) \\
&+ \frac{\partial}{\partial y}\left(\overline{w}\,\overline{v} + \overline{w}v^{'} + w^{'}\overline{v} + w^{'}v^{'}\right) \\
&+ \frac{\partial}{\partial z}\left(\overline{w}\,\overline{w} + \overline{w}w^{'} + w^{'}\overline{w} + w^{'}w^{'}\right) \\
&+ \frac{1}{\overline{\rho}}\frac{\partial p^{'}}{\partial z} + \frac{\rho^{'}}{\overline{\rho}} g \\
&- \nu\nabla^2\overline{w} - \nu\nabla^2 w^{'} = 0\,.
\end{aligned} \tag{19}$$

Now let's consider the term $\rho^{'}/\rho$. The equation of state is given by $p = \rho RT$, which is decomposed as

$$\frac{\overline{p}}{R} + \frac{p^{'}}{R} = \overline{\rho}\,\overline{T} + \overline{\rho}T^{'} + \rho^{'}\overline{T} + \rho^{'}T^{'}\,. \tag{20}$$

Next, we apply Reynolds averaging

$$\overline{\frac{\overline{p}}{R}} + \overline{\frac{p^{'}}{R}} = \overline{\overline{\rho}\,\overline{T}} + \overline{\overline{\rho}T^{'}} + \overline{\rho^{'}\overline{T}} + \overline{\rho^{'}T^{'}} \longrightarrow \frac{\overline{p}}{R} = \overline{\rho}\,\overline{T} + \overline{\rho^{'}T^{'}}.$$

Generally the last term is much smaller in magnitude than the others, which means it is safely neglected.

$$\frac{\overline{p}}{R} = \overline{\rho}\,\overline{T}\,. \tag{21}$$

Thus, the equation of state holds in the mean. This is reasonable because the equation of state was originally formulated from measurements made with crude, slow-response sensors. These sensors were essentially measuring mean quantities. (Stull 1988)

Next, we substract Eq.(21) from Eq.(20)

$$\frac{\overline{p}}{R} + \frac{p^{'}}{R} - \frac{\overline{p}}{R} = \overline{\rho}\,\overline{T} + \overline{\rho}T^{'} + \rho^{'}\overline{T} + \rho^{'}T^{'} - \overline{\rho}\,\overline{T} \longrightarrow \frac{p^{'}}{R} = \overline{\rho}T^{'} + \rho^{'}\overline{T} + \rho^{'}T^{'}.$$

Finally, divide this expression by Eq.(21)

$$\frac{p^{'}}{R}\frac{R}{\overline{p}} = \frac{\overline{\rho}T^{'}}{\overline{\rho}\,\overline{T}} + \frac{\rho^{'}\overline{T}}{\overline{\rho}\,\overline{T}} + \frac{\rho^{'}T^{'}}{\overline{\rho}\,\overline{T}} \longrightarrow \frac{p^{'}}{\overline{p}} = \frac{\rho^{'}}{\overline{\rho}} + \frac{T^{'}}{\overline{T}} + \frac{\rho^{'}}{\overline{\rho}}\frac{T^{'}}{\overline{T}}.$$

The last term is much smaller than the others is safely neglected, which leads to

$$\frac{p^{'}}{\overline{p}} = \frac{\rho^{'}}{\overline{\rho}} + \frac{T^{'}}{\overline{T}}\,. \tag{22}$$

Equation (22) is the linearized perturbation ideal gas law. Scale analysis shows that

$p^{'}/\overline{p} \rightarrow 0.1$ hPa/1000 hPa $= 10^{-4}$, while $T^{'}/\overline{T} \rightarrow 1$ K/300 K $= 3.33^{-3}$. Thus, we can neglect the pressure term, yielding

$$\frac{\rho^{'}}{\overline{\rho}} \approx -\frac{T^{'}}{\overline{T}}\,. \tag{23}$$

Recalling that $T^{'} = T - \overline{T}$ and making use of the Poisson equation, one can write

$$\frac{T^{'}}{\overline{T}} = \frac{\theta\left(\frac{p}{p_0}\right)^{\frac{R}{c_p}} - \overline{\theta}\left(\frac{\overline{p}}{p_0}\right)^{\frac{R}{c_p}}}{\overline{\theta}\left(\frac{\overline{p}}{p_0}\right)^{\frac{R}{c_p}}} = \frac{\theta\left(\frac{\overline{p}+p^{'}}{p_0}\right)^{\frac{R}{c_p}} - \overline{\theta}\left(\frac{\overline{p}}{p_0}\right)^{\frac{R}{c_p}}}{\overline{\theta}\left(\frac{\overline{p}}{p_0}\right)^{\frac{R}{c_p}}} \approx \frac{\theta\left(\frac{\overline{p}}{p_0}\right)^{\frac{R}{c_p}} - \overline{\theta}\left(\frac{\overline{p}}{p_0}\right)^{\frac{R}{c_p}}}{\overline{\theta}\left(\frac{\overline{p}}{p_0}\right)^{\frac{R}{c_p}}} \approx \frac{\theta - \overline{\theta}}{\overline{\theta}}.$$

Thus, one can use the approximation that

$$\frac{\rho^{'}}{\overline{\rho}} \approx -\frac{T^{'}}{\overline{T}} \approx -\frac{\theta^{'}}{\overline{\theta}}\,. \tag{24}$$

Using this relationship, Eq.(16) is rewritten in terms of buoyancy as

$$\begin{aligned}\frac{\partial \overline{w}}{\partial t}+\frac{\partial w^{'}}{\partial t}+\frac{\partial}{\partial x}&\left(\overline{w}\,\overline{u}+\overline{w}u^{'}+w^{'}\overline{u}+w^{'}u^{'}\right)\\&+\frac{\partial}{\partial y}\left(\overline{w}\,\overline{v}+\overline{w}v^{'}+w^{'}\overline{v}+w^{'}v^{'}\right)\\&+\frac{\partial}{\partial z}\left(\overline{w}\,\overline{w}+\overline{w}w^{'}+w^{'}\overline{w}+w^{'}w^{'}\right)\\&+\frac{1}{\overline{\rho}}\frac{\partial p^{'}}{\partial z}-\beta\theta^{'}\\&-\nu\nabla^2\overline{w}-\nu\nabla^2 w^{'}=0\,,\end{aligned}\tag{25}$$

where $\beta = g/\overline{\theta}$ is the buoyancy parameter. Before applying Reynolds averaging, we can rewrite the pressure term as

$$\frac{1}{\overline{\rho}}\frac{\partial p^{'}}{\partial z}=\frac{1}{\overline{\rho}}\frac{\partial p}{\partial z}-\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial z}=\frac{1}{\overline{\rho}}\frac{\partial p}{\partial z}+g.$$

Substituting this expression and taking the Reynolds average yields

$$\begin{aligned}\overline{\frac{\partial \overline{w}}{\partial t}}+\overline{\frac{\partial w^{'}}{\partial t}}+&\overline{\frac{\partial}{\partial x}(\overline{w}\,\overline{u}+\overline{w}u^{'}+w^{'}\overline{u}+w^{'}u^{'})}\\&+\overline{\frac{\partial}{\partial y}(\overline{w}\,\overline{v}+\overline{w}v^{'}+w^{'}\overline{v}+w^{'}v^{'})}\\&+\overline{\frac{\partial}{\partial z}(\overline{w}\,\overline{w}+\overline{w}w^{'}+w^{'}\overline{w}+w^{'}w^{'})}\\&+\overline{\frac{1}{\overline{\rho}}\frac{\partial p}{\partial z}}+\overline{g}-\overline{\beta\theta^{'}}\\&-\overline{\nu\nabla^2\overline{w}}-\overline{\nu\nabla^2 w^{'}}=0\,.\end{aligned}\tag{26}$$

Using a similar procedure as that for the horizontal momentum equations yields the vertical momentum balance equation, given by

$$\boxed{\frac{\partial \overline{w}}{\partial t}+\overline{u}\frac{\partial \overline{w}}{\partial x}+\overline{v}\frac{\partial \overline{w}}{\partial y}+\overline{w}\frac{\partial \overline{w}}{\partial z}+\frac{\partial \overline{w^{'}u^{'}}}{\partial x}+\frac{\partial \overline{w^{'}v^{'}}}{\partial y}+\frac{\partial \overline{w^{'}w^{'}}}{\partial z}+\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial z}+g-\beta\overline{\theta^{'}}-\nu\nabla^2\overline{w}=0}\,.\tag{27}$$

Note here that $\overline{\theta^{'}}$ is retained because it represents a departure from hydrostatic balance. Since the boundary layer is not assumed to be in hydrostatic balance, $\theta^{'}$ is not guaranteed to be zero.

Thermodynamic balance equation

Again we apply Reynolds decomposition and average the resulting expression

$$\begin{aligned}\overline{\frac{\partial\overline{\theta}}{\partial t}} + \overline{\frac{\partial\theta'}{\partial t}} + &\overline{\frac{\partial}{\partial x}\left(\overline{\theta}\,\overline{u} + \overline{\theta}u' + \theta'\overline{u} + \theta' u'\right)} \\ &+ \overline{\frac{\partial}{\partial y}\left(\overline{\theta}\,\overline{v} + \overline{\theta}v' + \theta'\overline{v} + \theta' v'\right)} \\ &+ \overline{\frac{\partial}{\partial z}\left(\overline{\theta}\,\overline{w} + \overline{\theta}w' + \theta'\overline{w} + \theta' w'\right)} \\ &+ \overline{\frac{1}{c_p\overline{\rho}}\left(\frac{\partial\overline{F_{rx}}}{\partial x} + \frac{\partial\overline{F_{ry}}}{\partial y} + \frac{\partial\overline{F_{rz}}}{\partial z}\right)} + \overline{\frac{1}{c_p\overline{\rho}}\left(\frac{\partial F'_{rx}}{\partial x} + \frac{\partial F'_{ry}}{\partial y} + \frac{\partial F'_{rz}}{\partial z}\right)} \\ &- \overline{\frac{1}{c_p\overline{\rho}}(\overline{H_{\text{so}}} - \overline{H_{\text{si}}})} - \overline{\frac{1}{c_p\overline{\rho}}(H'_{\text{so}} - H'_{\text{si}})} \\ &- \overline{\nu\nabla^2\overline{\theta}} - \overline{\nu\nabla^2\theta'} = 0.\end{aligned} \tag{28}$$

Applying the averaging rules results in the Reynolds-averaged thermodynamic energy equation, given by

$$\boxed{\frac{\partial\overline{\theta}}{\partial t} + \overline{u}\frac{\partial\overline{\theta}}{\partial x} + \overline{v}\frac{\partial\overline{\theta}}{\partial y} + \overline{\theta}\frac{\partial\overline{w}}{\partial z} + \frac{\partial\overline{\theta' u'}}{\partial x} + \frac{\partial\overline{\theta' v'}}{\partial y} + \frac{\partial\overline{\theta' w'}}{\partial z} + \overline{S_\theta} - \nu_h\nabla^2\overline{\theta} = 0} \tag{29}$$

Here $\overline{S_\theta}$ is the combined source/sink term that accounts for radiative flux and local heat sources.

Moisture balance equation

We split the terms into mean and fluctuating parts, which yields

$$\begin{aligned}\overline{\frac{\partial\overline{q}}{\partial t}} + \overline{\frac{\partial q'}{\partial t}} + &\overline{\frac{\partial}{\partial x}\left(\overline{q}\,\overline{u} + \overline{q}u' + q'\overline{u} + q' u'\right)} \\ &+ \overline{\frac{\partial}{\partial y}\left(\overline{q}\,\overline{v} + \overline{q}v' + q'\overline{v} + q' v'\right)} \\ &+ \overline{\frac{\partial}{\partial z}\left(\overline{q}\,\overline{w} + \overline{q}w' + q'\overline{w} + q' w'\right)} \\ &- \overline{\frac{1}{\overline{\rho}}(\overline{W_{\text{so}}} - \overline{W_{\text{si}}})} - \overline{\frac{1}{\overline{\rho}}(W'_{\text{so}} - W'_{\text{si}})} \\ &- \overline{\nu\nabla^2\overline{q}} - \overline{\nu\nabla^2 q'} = 0.\end{aligned} \tag{30}$$

The averaging procedure results in the moisture balance equation, given by

$$\boxed{\frac{\partial\overline{q}}{\partial t} + \overline{u}\frac{\partial\overline{q}}{\partial x} + \overline{v}\frac{\partial\overline{q}}{\partial y} + \overline{q}\frac{\partial\overline{w}}{\partial z} + \frac{\partial\overline{q' u'}}{\partial x} + \frac{\partial\overline{q' v'}}{\partial y} + \frac{\partial\overline{q' w'}}{\partial z} + \overline{S_q} - \nu_q\nabla^2\overline{q} = 0} \tag{31}$$

Here $\overline{S_q}$ is the combined source/sink term.