Mesoscale Meteorology

2.1.4 Flows over two-dimensional sinusoidal mountains (continued)

In the last class, we covered mountain-forced flows, reviewed general properties of internal gravity waves, and began considering the specific example of two-dimensional, steady-state, adiabatic, inviscid, non-rotating, Boussinesq fluid flow over two-dimensional sinusoidal mountains.

As part of our analysis, we derived Scorer's equation

$$\nabla^2 w' + l^2(z) w' = 0 \,, \tag{24}$$

where l is the Scorer parameter, which is defined by:

$$l(z) = \sqrt{\frac{N^2}{\overline{u}^2} - \frac{1}{\overline{u}}\frac{\partial^2 \overline{u}}{\partial z^2}} \,. \tag{25}$$

In the example of sinusoidal terrain, we also assumed that both the mean flow $\overline{u}(z)$ and stability $N(z)$ are both constant with height. In this scenario, l^2 reduces to $N^2/\overline{u}^2$. Also due to the sinusoidal terrain, we assumed that solutions to Eq (24) are of the form:

$$w' = \hat{w}(z) \exp(i[kx + mz]) \,, \tag{28}$$

where $k = 2\pi/L_x$ is the wavenumber of the terrain, L_x is the separation distance between ridges, $m = \sqrt{(l^2 - k^2)} = \sqrt{(N^2/\overline{u}^2 - k^2)}$, and $\hat{w}(z)$ is the wave amplitude described by the modified Taylor-Goldstein equation, defined as

$$\frac{\partial^2 \hat{w}}{\partial z^2} + m^2 \hat{w} = 0 \,. \tag{31}$$

Finally, we combined the solution to Eq. (31) with Eq. (28), and arrived at the following solution to w':

$$w' = A \exp\left(i[kx + mz]\right) + B \exp\left(i[kx - mz]\right) \,, \tag{38}$$

where A and B are constant complex amplitudes. Now we will examine the two possible cases from Eq.(38): when m is real and when m is imaginary.

m is real

Recall that m is defined by $\sqrt{(l^2 - k^2)}$. When m is real, then

$$l > k \quad \rightarrow \quad \frac{N}{\overline{u}} > k \quad \rightarrow \quad \frac{N}{\overline{u}} > \frac{2\pi}{L_x} \quad \rightarrow \quad NL_x > 2\pi\overline{u} \quad \rightarrow \quad \frac{(L_x/\overline{u})}{(2\pi/N)} > 1.$$

The numerator $(L_x/\overline{u})$, represents the advection time of an air parcel passing over one wavelength of the terrain, while the denominator $(2\pi/N)$ represents the period of buoyancy oscillation due to stratification. This means that the time an air parcel takes to pass over the terrain is more than it takes for vertical oscillation due to buoyancy force. In other words, buoyancy force plays a larger role than the horizontal advection.

Since m is real, we expect solutions to follow Eq. (38), where A and B must be determined. Recall that the top boundary condition for this problem states that waves must transport energy away from their energy source (avoid reflection). It can be shown that solution waves of the form $\exp(i[kx + mz])$ have an energy flux that is directed upward, while those of the form $\exp(i[kx - mz])$ have a downward energy flux. In order to satisfy our upper boundary condition, this implies $B = 0$, leaving a solution of the form

$$w' = A \exp(i[kx + mz]) = A_1 \cos(kx + mz) + A_2 \sin(kx + mz)\,. \tag{39}$$

To solve for A_1 and A_2, we invoke the lower boundary condition. Recall that from last class, the terrain behaves according to

$$h(x) = h_m \sin kx\,, \tag{29}$$

where h_m is mountain height. Then, because the flow at the lower boundary must be parallel to the boundary, the vertical velocity perturbation at the boundary is given by the rate at which the boundary height changes following the motion:

$$w'(x, 0) = \frac{Dh}{Dt}_{z=0} \approx \overline{u}\frac{\partial h}{\partial x} = \overline{u}kh_m \cos(kx) \tag{30}$$

.

Now we compare Eq. (30) and Eq. (39) at $z = 0$

$$w'(x, 0) = \overline{u}kh_m \cos(kx) = A_1 \cos(kx) + A_2 \sin(kx)\,. \tag{40}$$

This implies $A_1 = \overline{u}kh_m$ and $A_2 = 0$. Thus, the solution is

$$w' = \overline{u}kh_m \cos(kx + mz)\,. \tag{41}$$

This describes a wave that propagates vertically without loss of amplitude.

m is imaginary

When m is imaginary, then

$$l < k \quad \rightarrow \quad \frac{N}{\overline{u}} < k \quad \rightarrow \quad \frac{N}{\overline{u}} > \frac{2\pi}{L_x} \quad \rightarrow \quad NL_x < 2\pi\overline{u} \quad \rightarrow \quad \frac{(L_x/\overline{u})}{(2\pi/N)} < 1.$$

This means that the time an air parcel takes to pass over the terrain is less than it takes for vertical oscillation due to buoyancy force. In other words, buoyancy force plays a smaller role than the horizontal advection.

Since m is imaginary, we set $m = i\mu$, where $\mu = |m|$ is a real number. If we substitute this expression into Eq. (38), we arrive at our expected solution for the case where m is imaginary:

$$w^{'} = A\exp(ikx)\exp(-\mu z) + B\exp(ikx)\exp(\mu z)\ . \tag{42}$$

We first apply our upper boundary condition to help determine the coefficients. In this case, the B term represents a wave whose amplitude grows exponentially unbounded with height. This is unphysical since the energy source is the mountain surface. This implies that $B = 0$, leaving a solution of the form

$$w^{'} = A\exp(ikx)\exp(-\mu z) = A_1\exp(-\mu z)\cos(kx) + A_2\exp(-\mu z)\sin(kx)\ . \tag{43}$$

Now we compare Eq. (30) and Eq. (43) at $z = 0$

$$w^{'}(x,0) = \overline{u}kh_m\cos(kx) = A_1\cos(kx) + A_2\sin(kx)\ . \tag{44}$$

This again implies that $A_1 = \overline{u}kh_m$ and $A_2 = 0$. Thus, the solution is

$$w^{'} = \overline{u}kh_m\exp(-\mu z)\cos(kx)\ . \tag{45}$$

This describes a wave that decays exponentially with height.

Physical reasoning for the differences between the two cases

When m is real, it implies that $l > k$, or $N > \overline{u}k$. Here, $\overline{u}k$ is known as the *intrinsic frequency*, which is the frequency that a wave would have if observed in a reference frame moving with the mean wind. If we rearrange our previous definition of m to solve for $\overline{u}$, we get

$$\overline{u} = \pm\sqrt{\frac{N^2}{m^2 + k^2}}.$$

Using this expression, the inequality becomes

$$N > \pm \frac{Nk}{\sqrt{m^2 + k^2}}.$$

When m is real, the total wave number may be regarded as a vector, directed perpendicular to lines of constant phase, and in the direction of phase increase. In this case, the wavenumber vector is tilted at an angle ϕ relative to vertical, with a horizontal component of k and a vertical component of m. Thus, it is trivial to show that $k/\sqrt{m^2 + k^2} = \cos(\phi)$. Accordingly, the inequality becomes

$$N > N \cos \phi.$$

With buoyancy as the restoring force, the atmosphere can support oscillations with frequencies less than or equal to N for angles with respect to the vertical varying between $90\,\mathrm{deg}$ (purely horizontal) and $0\,\mathrm{deg}$ (purely vertical). To relate this to the mountain wave problem, we must realize that the flow over the terrain is driving an oscillation at a frequency with a magnitude equal to $\overline{u}k$. As long as this frequency is less than N we can find a slanted path along which the oscillation can be supported. Once the frequency exceeds N, there is no real angle ϕ that satisfies the inequality. Thus, no such slanted path is possible and the waves simply decay with height, which is the case where m is imaginary.

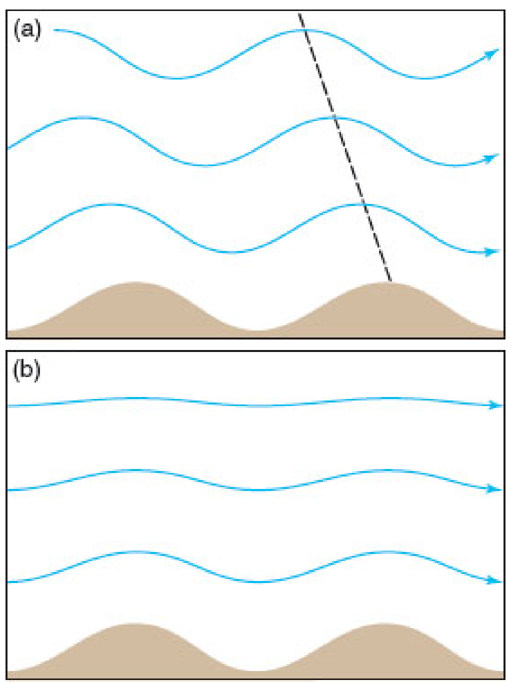

Figure 1: Streamlines in steady flow over an infinite series of sinusoidal ridges (a) for the case where $l > k$ and (b) for the case where $l < k$. The dashed line in (a) shows the phase of maximum upward displacement, which tilts westward with height. [From Markowski and Richardson]

Wave types partitioned by Scorer parameter

We have found that when $l > k$, waves forced by sinusoidal terrain are tilted and vertically propagating, while $l < k$ results in evanescent waves. Now we consider the limits in both directions. As covered in the last class, when $l \ll k$, advection dominates buoyancy and Eq. (24) reduces to

$$\nabla^2 w' = 0 \,, \tag{26}$$

which represents irrotational or potential flow.

Conversely, when $l \gg k$, buoyancy dominates advection and the governing equation becomes

$$\nabla^2 w' + l^2 w' = 0 \,. \tag{46}$$

In other words, the vertical pressure gradient force and the buoyancy force are roughly in balance and the vertical acceleration can be ignored. Thus, the mountain waves become hydrostatic. Here, the flow pattern repeats itself in the vertical with a wavelength of $L_z = 2\pi/l = 2\pi\overline{u}/N$, which is also referred to as the hydrostatic vertical wavelength.

The boundary between the regimes of vertically propagating waves and evanescent waves can be found by letting $l = k$, which leads to $L = 2\pi\overline{u}/N$.

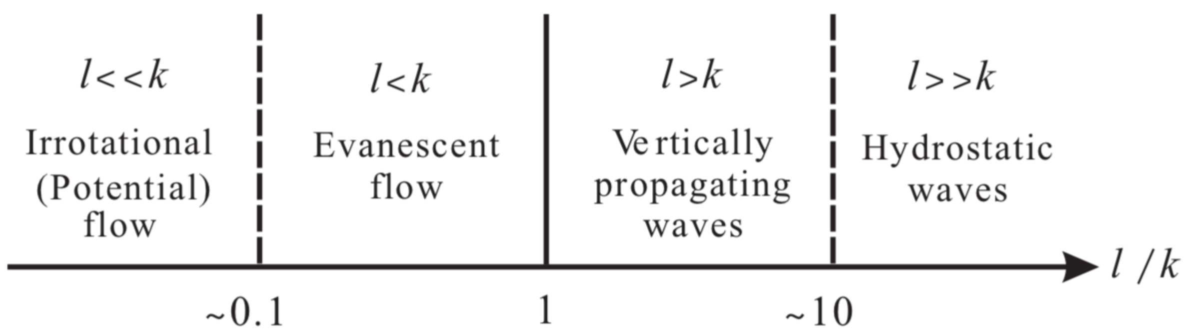

Figure 2: Relations among different mountain waves as determined by l/k, where l is the Scorer parameter and k is the wave number.. [From Lin (2010)]

2.1.5 Flows over two-dimensional isolated mountains

Earlier, we obtained mountain wave solution forced by a sinusoidal orography. However, an endless series of ridges is not a common form of topography. More often, the topography consists of an isolated mountain or a single mountain chain, approximated as a single two-dimensional ridge. In this section, we extend the results of the previous section to such an isolated ridge. To do this, we perform a Fourier transform on the real orography, therefore any orography can be considered summation of many sinusoidal modes or wave components.

When the orography is low, the wave solutions are nearly linear, therefore the total solution is sum of all forced waves.

Let $\hat{w}$ be the amplitude of the wave component with wavenumber k, Eq. (31) we obtained earlier is expanded

$$\frac{\partial^2 \hat{w}}{\partial z^2} + (l^2 - k^2)\hat{w} = 0 \, . \tag{47}$$

The Fourier transform of the linear lower boundary condition is

$$\hat{w}(k, z = 0) = ik\overline{u}\hat{h}(k) \, , \tag{48}$$

where $\hat{h}(k)$ is the amplitude of Fourier component of orography with wavenumber k. For constant Scorer parameter, the solution we obtained earlier for the wave amplitude can be written into two parts,

$$\hat{w}(k, z = 0) = \begin{cases} \hat{w}(k, 0)\exp(i\sqrt{l^2 - k^2}z) & \text{if } l^2 > k^2, \text{ and} \quad (49\text{a}) \\ \hat{w}(k, 0)\exp(-\sqrt{k^2 - l^2}z) & \text{if } l^2 < k^2. \quad (49\text{b}) \end{cases}$$

Taking the inverse one-sided Fourier transform of Eq. (49) yields the solution in the physical space,

$$\begin{aligned} w^{'}(x, z) =& 2\Re\left[\int_0^l ik\overline{u}\hat{h}(k)\exp(i\sqrt{l^2 - k^2}z)\exp(ikx)dk \right. \\ & \left. + \int_l^\infty ik\overline{u}\hat{h}(k)\exp(-\sqrt{k^2 - l^2}z)\exp(ikx)dk\right] \, , \end{aligned} \tag{50}$$

which is basically the summation of all wave components. The first integration represents the upward propagating wave, while the second integration represents the evanescent waves.

For simplicity, let us consider a bell-shaped mountain mountain profile. One commonly assumed profile is the ‘Witch of Agnesi’ profile, given by

$$h(x) = \frac{h_m a^2}{x^2 + a^2} \, , \tag{51}$$

where h_m is the mountain height and a is the half-width. The mountain peak is at $x = 0$. This profile has the desirable properties of asymptotically approaching zero as $x \to \infty$ and having a width that is easily tuned by varying a, the shape parameter. The horizontal wavelengths associated with the terrain will be determined by the value of a, with longer wavelengths (*i.e.*, wider mountains) for larger values of a. Thus, for the same environmental conditions, we can anticipate that the larger (smaller) we make a, the more (less) likely the solution will contain wavenumbers that satisfy $N > \overline{u}k$, such that waves propagate vertically.

The one-sided Fourier transform of this mountain profile is in a simple form

$$\hat{h}(k) = \frac{h_m a}{2} \exp\left(-ka\right), \text{ for } k > 0. \tag{52}$$

This amplitude equation is plugged into Eq. (50) to solve for $w^{'}$. For this bell shaped mountain, the characteristic wavenumber of forcing is $k = 1/a$. We now consider three possible cases.

$l^2 \ll k^2$

Again, we assumed that $\overline{u}$ and N are constant with height. In this case, the second integral on the right hand side of Eq. (50) can be neglected, and the final solutions are

$$w'(x,z) \approx 2\Re\left[\overline{u}\int_0^\infty ik\left(\frac{h_m a}{2}\right)\exp\left(-ka\right)\exp\left(-kz\right)\exp\left(ikx\right)dk\right] \tag{53}$$

Therefore, similar to the sinusoidal mountain case, the flow pattern is symmetric with respect to the center of the mountain ridge ($x = 0$). However, the amplitude decreases with height linearly, instead of exponentially. The flow pattern is depicted in Fig. 3.

$l^2 \gg k^2$

In this case, the first integral on the right hand side of Eq. (50) can be neglected, and the final solutions are

$$w'(x,z) \approx 2\Re\left[\overline{u}\int_0^\infty ik\hat{h}(k)\exp\left(ilz\right)\exp\left(ikx\right)dk\right] \tag{54}$$

$$= 2\mathrm{Re}\left[\overline{u}\int_0^\infty ik\left(\frac{h_m a}{2}\right)\exp\left(-ka\right)\exp\left(ilz\right)\exp\left(ikx\right)dk\right]. \tag{55}$$

This type of flow is characterized as a hydrostatic mountain wave. The disturbance confines itself over the mountain in horizontal, but repeats itself in vertical with a wavelength of $2\pi\overline{u}/N$.

$l^2 \approx k^2$

In this case, all terms in Eq. (50) are equally important. Both asymptotic methods and numerical methods can be used to solve the problem. The solution is shown in Fig. 4

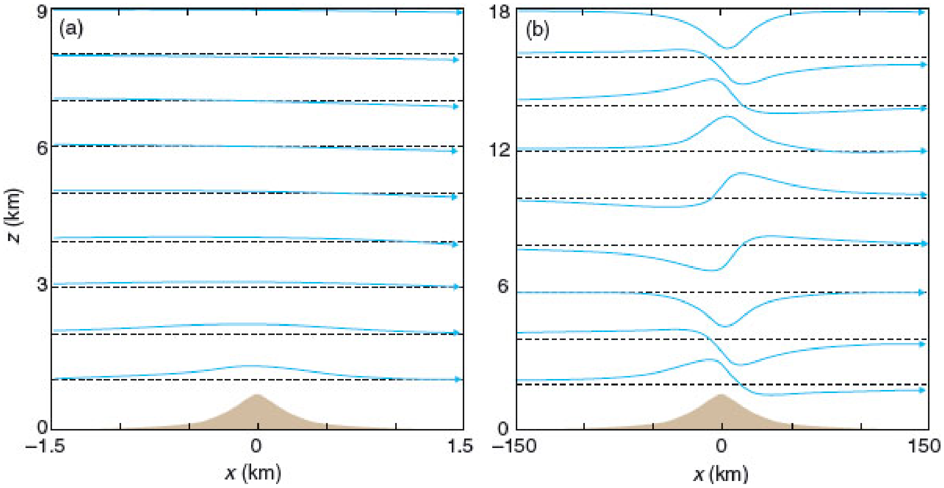

Figure 3: Streamlines in steady airflow over an isolated ridge when (a) $\overline{u}a - 1 \ll N$ and (b) $\overline{u}a - 1 \gg N$. [From Markowski and Richardson]

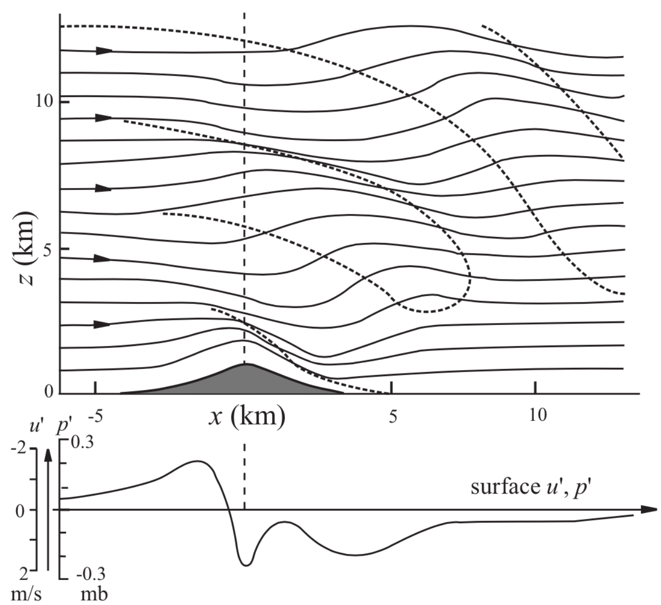

Figure 4: Flow over a two-dimensional ridge of intermediate width ($l^2 \approx k^2$) where the buoyancy force is important, but not so dominant that the flow is hydrostatic. The zero phase lines are denoted by dotted curves. The waves on the lee aloft are the dispersive tail of the nonhydrostatic waves ($k < l$ but not $k << l$). The flow and orographic parameters are: $U = 10\text{ms}^{-1}$, $N = 0.01\text{s}^{-1}$, $h_m = 1\text{km}$, and $a = 1\text{km}$. (Adapted after Queney 1948)

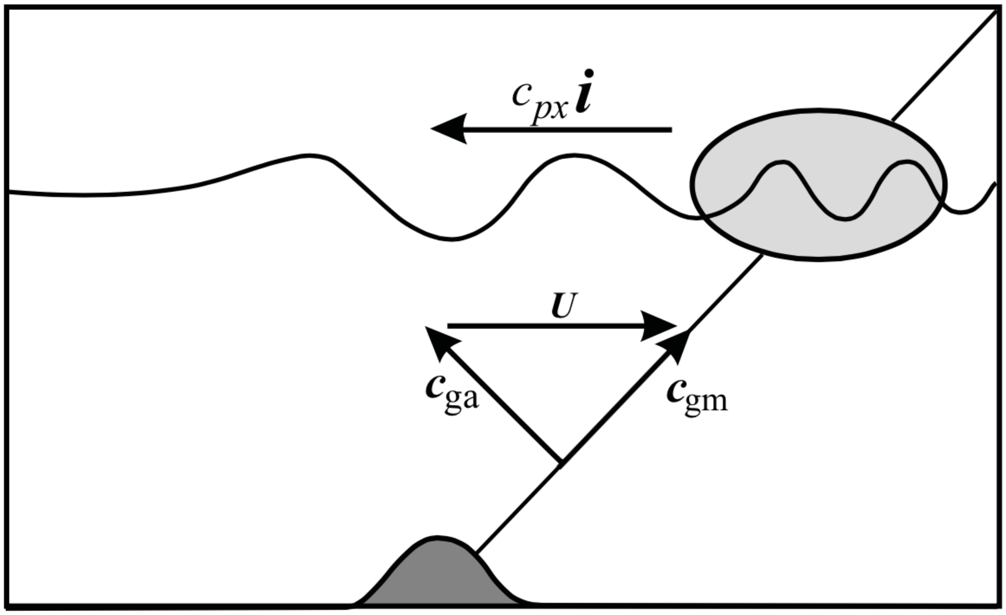

Figure 5: A schematic illustrating the relationship among the group velocity with respect to (w.r.t) to the air (c_{ga}), group velocity w.r.t. to the mountain (c_{gm}), horizontal phase speed ($c_{px}\boldsymbol{i}$) and the basic wind. The horizontal phase speed of the wave is exactly equal and opposite to the basic wind speed. The wave energy propagates upward and upstream relative to the air, but is advected downstream by the basic wind. The energy associated with the mountain waves propagates upward and downstream relative to the mountain. (Adapted after Smith 1979)