7.4 INTERNAL GRAVITY (BUOYANCY) WAVES

We now consider the nature of gravity wave propagation in the atmosphere. Atmospheric gravity waves can only exist when the atmosphere is stably stratified so that a fluid parcel displaced vertically will undergo buoyancy oscillations (see Section 2.7.3). Because the buoyancy force is the restoring force responsible for gravity waves, the term *buoyancy wave* is actually more appropriate as a name for these waves. However, in this text we will generally use the traditional name *gravity wave*.

In a fluid, such as the ocean, which is bounded both above and below, gravity waves propagate primarily in the horizontal plane since vertically traveling waves are reflected from the boundaries to form standing waves. However, in a fluid that has no upper boundary, such as the atmosphere, gravity waves may propagate vertically as well as horizontally. In vertically propagating waves the phase is a function of height. Such waves are referred to as *internal* waves. Although internal gravity waves are not generally of great importance for synoptic-scale weather forecasting (and indeed are nonexistent in the filtered quasi-geostrophic models), they can be important in mesoscale motions. For example, they are responsible for the occurrence of mountain *lee waves*. They also are believed to be an important mechanism for transporting energy and momentum into the middle atmosphere, and are often associated with the formation of clear air turbulence (CAT).

7.4.1 Pure Internal Gravity Waves

For simplicity we neglect the Coriolis force and limit our discussion to two-dimensional internal gravity waves propagating in the x, z plane. An expression for the frequency of such waves can be obtained by modifying the parcel theory developed in Section 2.7.3.

Internal gravity waves are transverse waves in which the parcel oscillations are parallel to the phase lines as indicated in Fig. 7.8. A parcel displaced a distance δs along a line tilted at an angle α to the vertical as shown in Fig. 7.8 will undergo a vertical displacement $\delta z = \delta s \cos\alpha$. For such a parcel the *vertical* buoyancy force per unit mass is just $-N^2\delta z$, as was shown in (2.52). Thus, the component of the buoyancy force parallel to the tilted path along which the parcel oscillates is just

$$-N^2\delta z\cos\alpha = -N^2\left(\delta s\cos\alpha\right)\cos\alpha = -\left(N\cos\alpha\right)^2\delta s$$

The momentum equation for the parcel oscillation is then

$$\frac{d^2\left(\delta s\right)}{dt^2} = -\left(N\cos\alpha\right)^2\delta s \tag{7.24}$$

which has the general solution $\delta s = \exp\left[\pm i\left(N\cos\alpha\right)t\right]$. Thus, the parcels execute a simple harmonic oscillation at the frequency $\nu = N\cos\alpha$. This frequency

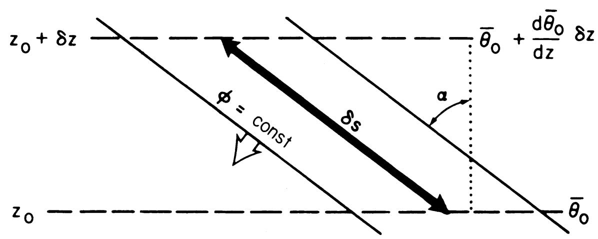

Fig. 7.8 Parcel oscillation path (heavy arrow) for pure gravity waves with phase lines tilted at an angle α to the vertical.

depends only on the static stability (measured by the buoyancy frequency N) and the angle of the phase lines to the vertical.

The above heuristic derivation can be verified by considering the linearized equations for two-dimensional internal gravity waves. For simplicity, we employ the *Boussinesq approximation*, in which density is treated as a constant except where it is coupled with gravity in the buoyancy term of the vertical momentum equation. Thus, in this approximation the atmosphere is considered to be incompressible, and local density variations are assumed to be small perturbations of the constant basic state density field. Because the vertical variation of the basic state density is neglected except where coupled with gravity, the Boussinesq approximation is only valid for motions in which the vertical scale is less than the atmospheric scale height $H(\approx 8km)$.

Neglecting effects of rotation, the basic equations for two-dimensional motion of an incompressible atmosphere may be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{7.25}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0 \tag{7.26}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{7.27}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = 0 \tag{7.28}$$

where the potential temperature θ is related to pressure and density by

$$\theta = \frac{p}{\rho R}\left(\frac{p_s}{p}\right)^{\kappa}$$

which after taking logarithms on both sides yields

$$\ln\theta = \gamma^{-1}\ln p - \ln\rho + \text{constant} \tag{7.29}$$

We now linearize (7.25)–(7.29) by letting

$$\begin{aligned} \rho &= \rho_0 + \rho' & u &= \overline{u} + u' \\ p &= \overline{p}(z) + p' & w &= w' \\ \theta &= \overline{\theta}(z) + \theta' \end{aligned} \tag{7.30}$$

where the basic state zonal flow $\overline{u}$ and the density ρ_0 are both assumed to be constant. The basic state pressure field must satisfy the hydrostatic equation

$$d\overline{p}/dz = -\rho_0 g \tag{7.31}$$

while the basic state potential temperature must satisfy (7.29) so that

$$\ln \overline{\theta} = \gamma^{-1} \ln \overline{p} - \ln \rho_0 + \text{constant} \tag{7.32}$$

The linearized equations are obtained by substituting from (7.30) into (7.25)–(7.29) and neglecting all terms that are products of the perturbation variables. Thus, for example, the last two terms in (7.26) are approximated as

$$\begin{aligned} \frac{1}{\rho}\frac{\partial p}{\partial z} + g &= \frac{1}{\rho_0 + \rho'}\left(\frac{d\overline{p}}{dz} + \frac{\partial p'}{\partial z}\right) + g \\ &\approx \frac{1}{\rho_0}\frac{d\overline{p}}{dz}\left(1 - \frac{\rho'}{\rho_0}\right) + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} + g = \frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \frac{\rho'}{\rho_0} g \end{aligned} \tag{7.33}$$

where (7.31) has been used to eliminate $\overline{p}$. The perturbation form of (7.29) is obtained by noting that

$$\ln\left[\overline{\theta}\left(1 + \frac{\theta'}{\overline{\theta}}\right)\right] = \gamma^{-1} \ln\left[\overline{p}\left(1 + \frac{p'}{\overline{p}}\right)\right] - \ln\left[\rho_0\left(1 + \frac{\rho'}{\rho_0}\right)\right] + \text{const.} \tag{7.34}$$

Now, recalling that $\ln(ab) = \ln(a) + \ln(b)$ and that $\ln(1 + \varepsilon) \approx \varepsilon$ for any $\varepsilon \ll 1$, we find with the aid of (7.32) that (7.34) may be approximated by

$$\frac{\theta'}{\overline{\theta}} \approx \frac{1}{\gamma}\frac{p'}{\overline{p}} - \frac{\rho'}{\rho_0}$$

Solving for ρ' yields

$$\rho' \approx -\rho_0 \frac{\theta'}{\overline{\theta}} + \frac{p'}{c_s^2} \tag{7.35}$$

where $c_s^2 \equiv \overline{p}\gamma/\rho_0$ is the square of the speed of sound. For buoyancy wave motions $\left|\rho_0 \theta'/\overline{\theta}\right| \gg \left|p'/c_s^2\right|$; that is, density fluctuations due to pressure changes are small compared with those due to temperature changes. Therefore, to a first approximation,

$$\theta' / \overline{\theta} = -\rho' / \rho_0 \tag{7.36}$$

Using (7.33) and (7.36), the linearized version of the set (7.25)–(7.28), we can write as

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right) u' + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = 0 \tag{7.37}$$

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right) w' + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\theta'}{\overline{\theta}} g = 0 \tag{7.38}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \tag{7.39}$$

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right) \theta' + w'\frac{d\overline{\theta}}{dz} = 0 \tag{7.40}$$

Subtracting $\partial(7.37)/\partial z$ from $\partial(7.38)/\partial x$, we can eliminate p' to obtain

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right) - \frac{g}{\overline{\theta}}\frac{\partial \theta'}{\partial x} = 0 \tag{7.41}$$

which is just the y component of the vorticity equation.

With the aid of (7.39) and (7.40), u' and θ' can be eliminated from (7.41) to yield a single equation for w':

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) + N^2 \frac{\partial^2 w'}{\partial x^2} = 0 \tag{7.42}$$

where $N^2 \equiv g\, d \ln \overline{\theta}/dz$ is the square of the buoyancy frequency, which is assumed to be constant.[5]

Equation (7.42) has harmonic wave solutions of the form

$$w' = \text{Re}\left[\hat{w} \exp(i\phi)\right] = w_r \cos\phi - w_i \sin\phi \tag{7.43}$$

where $\hat{w} = w_r + i w_i$ is a complex amplitude with real part w_r and imaginary part w_i, and $\phi = kx + mz - \nu t$ is the phase, which is assumed to depend linearly on z as well as on x and t. Here the horizontal wave number k is real because the solution is always sinusoidal in x. The vertical wave number $m = m_r + i m_i$ may,

[5] Strictly speaking, N^2 cannot be exactly constant if ρ_0 is constant. However, for shallow disturbances the variation of N^2 with height is unimportant.

however, be complex, in which case m_r describes sinusoidal variation in z and m_i describes exponential decay or growth in z depending on whether m_i is positive or negative. When m is real, the total wave number may be regarded as a vector $\boldsymbol{\kappa} \equiv (k, m)$, directed perpendicular to lines of constant phase, and in the direction of phase increase, whose components, $k = 2\pi/L_x$ and $m = 2\pi/L_z$, are inversely proportional to the horizontal and vertical wavelengths, respectively. Substitution of the assumed solution into (7.42) yields the dispersion relationship

$$(\nu - \overline{u}k)^2 \left(k^2 + m^2\right) - N^2 k^2 = 0$$

so that

$$\hat{\nu} \equiv \nu - \overline{u}k = \pm Nk / \left(k^2 + m^2\right)^{1/2} = \pm Nk / |\boldsymbol{\kappa}| \tag{7.44}$$

where $\hat{\nu}$, the *intrinsic frequency*, is the frequency relative to the mean wind. Here, the plus sign is to be taken for eastward phase propagation and the minus sign for westward phase propagation, relative to the mean wind.

If we let $k > 0$ and $m < 0$, then lines of constant phase tilt eastward with increasing height as shown in Fig. 7.9 (i.e., for $\phi = kx + mz$ to remain constant as x increases, z must also increase when $k > 0$ and $m < 0$). The choice of the positive root in (7.44) then corresponds to eastward and downward phase propagation relative to the mean flow with horizontal and vertical phase speeds (relative to the

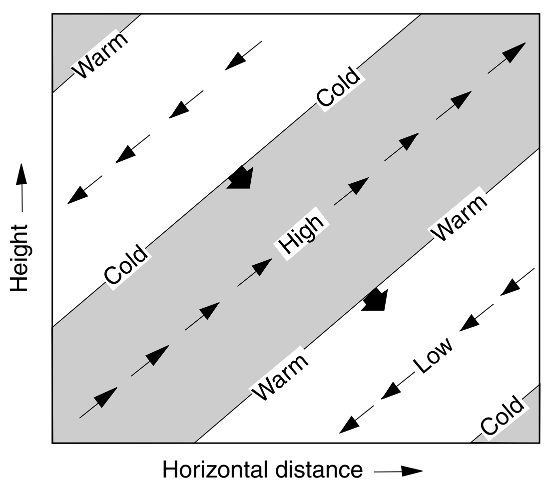

Fig. 7.9 Idealized cross section showing phases of pressure, temperature, and velocity perturbations for an internal gravity wave. Thin arrows indicate the perturbation velocity field, blunt solid arrows the phase velocity. Shading shows regions of upward motion.

mean flow) given by $c_x = \hat{\nu}/k$ and , $c_z = \hat{\nu}/m$ respectively.[6] The components of the group velocity, c_{gx} and c_{gz}, however, are given by

$$c_{gx} = \frac{\partial \nu}{\partial k} = \overline{u} \pm \frac{Nm^2}{\left(k^2 + m^2\right)^{3/2}} \tag{7.45a}$$

$$c_{gz} = \frac{\partial \nu}{\partial m} = \pm \frac{(-Nkm)}{\left(k^2 + m^2\right)^{3/2}} \tag{7.45b}$$

where the upper or lower signs are chosen in the same way as in (7.44). Thus, the vertical component of group velocity has a sign opposite to that of the vertical phase speed relative to the mean flow (downward phase propagation implies upward energy propagation). Furthermore, it is easily shown from (7.45) that the group velocity vector is parallel to lines of constant phase. Internal gravity waves thus have the remarkable property that group velocity is perpendicular to the direction of phase propagation. Because energy propagates at the group velocity this implies that energy propagates parallel to the wave crests and troughs, rather than perpendicular to them as in acoustic waves or shallow water gravity waves. In the atmosphere, internal gravity waves generated in the troposphere by cumulus convection, by flow over topography, and by other processes may propagate upward many scale heights into the middle atmosphere, even though individual fluid parcel oscillations may be confined to vertical distances much less than a kilometer.

Referring again to Fig. 7.9 it is evident that the angle of the phase lines to the local vertical is given by

$$\cos\alpha = L_z/(L_x^2 + L_z^2)^{1/2} = \pm k/(k^2 + m^2)^{1/2} = \pm k/\left|\boldsymbol{\kappa}\right|$$

Thus, $\hat{\nu} = \pm N \cos\alpha$ (i.e., gravity wave frequencies must be less than the buoyancy frequency) in agreement with the heuristic parcel oscillation model (7.24). The tilt of phase lines for internal gravity waves depends only on the ratio of the intrinsic wave frequency to the buoyancy frequency, and is independent of wavelength.

7.4.2 Topographic Waves

When air with mean wind speed $\overline{u}$ is forced to flow over a sinusoidal pattern of ridges under statically stable conditions, individual air parcels are alternately displaced upward and downward from their equilibrium levels and will thus undergo buoyancy oscillations as they move across the ridges as shown in Fig. 7.10. In

[6] Note that phase speed is not a vector. The phase speed in the direction perpendicular to constant phase lines (i.e., the blunt arrows in Fig. 7.9) is given by $\nu/(k^2 + m^2)^{1/2}$, which is not equal to $(c_x^2 + c_z^2)^{1/2}$.

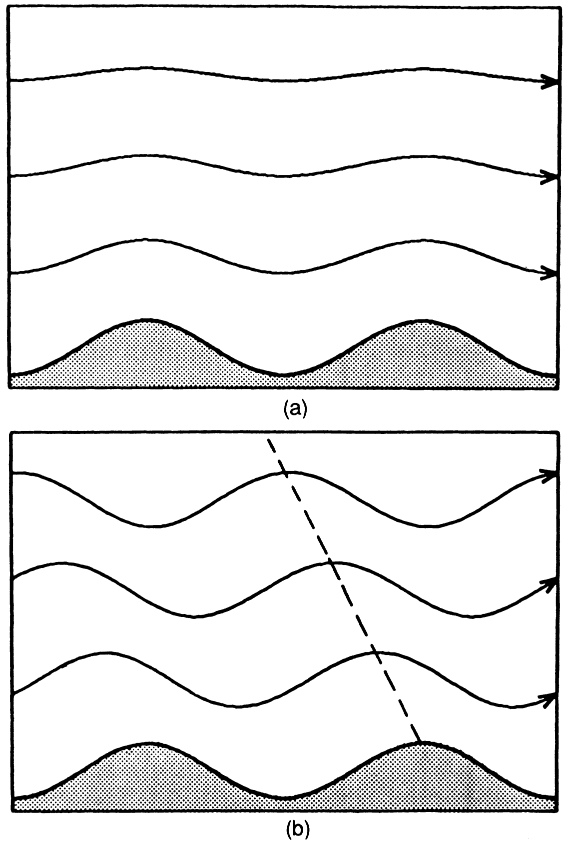

Fig. 7.10 Streamlines in steady flow over an infinite series of sinusoidal ridges for the narrow ridge case (a) and broad ridge case (b). The dashed line in (b) shows the phase of maximum upward displacement. (After Durran, 1990.)

this case there are solutions in the form of waves that are stationary relative to the ground [i.e, $\nu = 0$ in (7.43)]. For such stationary waves, w' depends only on (x, z) and (7.42) simplifies to

$$\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) + \frac{N^2}{\overline{u}^2} w' = 0 \tag{7.46}$$

Substituting from (7.43) into (7.46) then yields the dispersion relationship

$$m^2 = N^2 \Big/ \overline{u}^2 - k^2 \tag{7.47}$$

For given values of N, k, and $\overline{u}$, (7.47) determines the vertical structure. Then if $|\overline{u}| < N/k$, (7.47) shows that $m^2 > 0$ (i.e., m must be real) and solutions of (7.46) have the form of vertically propagating waves:

$$w' = \hat{w} \exp\left[i\left(kx + mz\right)\right]$$

Here we see from (7.44) that if we set $k > 0$ then for $\overline{u} > 0$ we have $\hat{\nu} < 0$ so that $m > 0$, whereas for $\overline{u} < 0$ we have $m < 0$. In the former situation the lower signs apply on the right in (7.45a,b), whereas in the latter the upper signs apply. In both cases the vertical phase propagation is downward relative to the mean flow, and vertical energy propagation is upward.

When $m^2 < 0, m = im_i$ is imaginary and the solution to (7.46) will have the form of vertically trapped waves:

$$w' = \hat{w} \exp(ikx) \exp(-m_i z)$$

Thus, vertical propagation is possible only when $|\overline{u}k|$, the magnitude of the frequency relative to the mean flow, is less than the buoyancy frequency. Stable stratification, wide ridges, and comparatively weak zonal flow provide favorable conditions for the formation of vertically propagating topographic waves (m real). Because the energy source for these waves is at the ground, they must transport energy upward. Hence, the phase speed relative to the mean zonal flow must have a downward component. Thus if $\overline{u} > 0$, lines of constant phase must tilt westward with height. When m is imaginary, however, the solution (7.43) has exponential behavior in the vertical with an exponential decay height of μ^{-1}, where $\mu = |m|$. Boundedness as $z \to \infty$ requires that we choose the solution with exponential decay away from the lower boundary.

In order to contrast the character of the solutions for real and imaginary m, we consider a specific example in which there is westerly mean flow over topography with a height profile given by

$$h(x) = h_M \cos kx$$

where h_M is the amplitude of the topography. Then because the flow at the lower boundary must be parallel to the boundary, the vertical velocity perturbation at the boundary is given by the rate at which the boundary height changes following the motion:

$$w'(x, 0) = \left(Dh/Dt\right)_{z=0} \approx \overline{u}\partial h/\partial x = -\overline{u}kh_M \sin kx$$

and the solution of (7.46) that satisfies this condition can be written

$$w(x, z) = \left\{ \begin{array}{ll} -\overline{u}h_M k e^{-\mu z} \sin kx, & \overline{u}k > N \\ -\overline{u}h_M k \sin(kx + mz), & \overline{u}k < N \end{array} \right\} \tag{7.48}$$

For fixed mean wind and buoyancy frequency, the character of the solution depends only on the horizontal scale of the topography. The two cases of (7.48) may be regarded as narrow ridge and wide ridge cases, respectively, since for specified values of $\overline{u}$ and N the character of the solution is determined by the zonal wave number k. The streamline patterns corresponding to these cases for westerly flow are illustrated in Fig. 7.10. In the narrow ridge case (Fig. 7.10a), the maximum upward displacement occurs at the ridge tops, and the amplitude of the disturbance decays with height. In the wide ridge case (Fig. 7.10b), the line of maximum upward displacement tilts back toward the west ($m > 0$), and amplitude is independent of height consistent with an internal gravity wave propagating westward relative to the mean flow.

Alternatively, for fixed zonal wave number and buoyancy frequency the solution depends only on the speed of the mean zonal wind. As indicated in (7.48), only for mean zonal wind magnitudes less than the critical value N/k will vertical wave propagation occur.

Equation (7.46) was obtained for conditions of constant basic state flow. In reality, both the zonal wind $\overline{u}$ and the stability parameter N generally vary with height, and ridges are usually isolated rather than periodic. A wide variety of responses are possible depending on the shape of the terrain and wind and stability profiles. Under certain conditions, large-amplitude waves can be formed, which may generate severe downslope surface winds and zones of strong clear air turbulence. Such circulations are discussed further in Section 9.4.

7.5 GRAVITY WAVES MODIFIED BY ROTATION

Gravity waves with horizontal scales greater than a few hundred kilometers and periods greater than a few hours are hydrostatic, but they are influenced by the Coriolis effect and are characterized by parcel oscillations that are elliptical rather than straight lines as in the pure gravity wave case. This *elliptical polarization* can be understood qualitatively by observing that the Coriolis effect resists horizontal parcel displacements in a rotating fluid, but in a manner somewhat different from that in which the buoyancy force resists vertical parcel displacements in a statically stable atmosphere. In the latter case the resistive force is opposite to the direction of parcel displacement, whereas in the former it is at right angles to the horizontal parcel velocity.

7.5.1 Pure Inertial Oscillations

Section 3.2.3 showed that a parcel put into horizontal motion in a resting atmosphere with constant Coriolis parameter executes a circular trajectory in an anticyclonic sense. A generalization of this type of inertial motion to the case with a geostrophic mean zonal flow can be derived using a parcel argument similar to that used for the buoyancy oscillation in Section 2.7.3.