Meteorology 5344, Fall 2017
Computational Fluid Dynamics
Instructor: Dr. Ming Xue

Computer Problem #3 Accuracy of Finite Differences
Distributed September 28, 2017
Due October 17, 2017

1. Consider the analytic function $f(x) = sin(\pi x/2)$. The goal of this exercise is to compare the analytical values of df/dx at a point with those obtained using the following numerical approximations:

- Centered Difference: $\dfrac{f_{i+1} - f_{i-1}}{2\Delta x}$

- Forward Difference: $\dfrac{f_{i+1} - f_i}{\Delta x}$

- Backward difference: $\dfrac{f_i - f_{i-1}}{\Delta x}$

- 3-point Asymmetric: $\dfrac{-1.5 f_i + 2 f_{i+1} - 0.5 f_{i+2}}{\Delta x}$

- 4-point Symmetric: $\dfrac{f_{i-2} - 8 f_{i-1} + 8 f_{i+1} - f_{i+2}}{12\Delta x}$

a) Evaluate df/dx at $x = 0.5$ using the above 5 formulae with a step size $\Delta x = 0.1$, and compute the error of your discretization scheme relative to the analytical solution, i.e.,

$$\varepsilon_r^d = \frac{\left(\dfrac{df}{dx}\right)_{discrete} - \left(\dfrac{df}{dx}\right)_{analytical}}{\left(\dfrac{df}{dx}\right)_{analytical}}. \qquad (1)$$

Write a computer program for this.

b). For each of the above five schemes, derive an expression for the truncation error τ and evaluate analytically the leading term at $x = 0.5$ and use it to estimate the numerical error. Calculate the corresponding estimated relative error and compare it with that obtain in a) above for each scheme. Present your results in **tabular form** and discuss the results.

Please note that our definition of the truncation error is effectively

$$\tau = \left(\frac{df}{dx}\right)_{analytical} - \left(\frac{df}{dx}\right)_{discrete}$$ so there is a sign difference between τ and the absolute

error used in (1). You should compare $-\tau$ with the numerical error.

c) Repeat part a) using $\Delta x = 0.5, 0.25, 0.125, 0.1, 0.05, 0.025, 0.0125$ and compute the error for each Δx and plot the results on a graph using $- log_{10}| \text{Error} |$ as the vertical axis and $- log_{10}(\Delta x)$ as the horizontal axis. The slope of this line is the convergence rate or actual order of accuracy of the scheme. How does this rate compare with that predicted by the leading term of the truncation error? (Note: Double precision may be required for the last two schemes. Consult the compiler man page to the compiler option for double precision. On Boomer, for ifort compiler, the option is '-r8'. See ifort man pages from 'man ifort'.).

2. Consider the function $f(x) = tanh[k (x-1)]$ where x and k are positive numbers.

a) Plot this function for $k = 1.0, 5.0$ and 20.0 in a domain $x \in [0.0, 2.0]$ using $\Delta x = 0.01$. Note the structure of the function for different values of k.

b). Compute df/dx at $x = 1.05$ using all five schemes from problem 1 with $\Delta x = 0.32, 0.16, 0.08, 0.04, 0.02, 0.01$ for both $k=5.0$ and $k=20.0$. Compute the error of the approximation as in part c) of problem 1 and plot it on a graph using $- log_{10}| \text{Error} |$ as the vertical axis and $- log_{10}(\Delta x)$ as the horizontal axis.

c) Discuss the nature of this error with changing Δx and k. What do the results suggest about the influence of a particular function on the truncation error? Do higher order schemes always give better results?