Chapter 3. Finite Difference Methods for Hyperbolic Equations

3.1. Introduction

Most hyperbolic problems involve the transport of fluid properties. In the equations of motion, the term describing the transport process is often called convection or advection.

E.g., the 1-D equation of motion is

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \,. \tag{1}$$

Here the advection term $u\dfrac{\partial u}{\partial x}$ term is nonlinear.

We will focus first on linear advection problem, and move to nonlinear problems later.

From (1), we can see the transport process can be expressed in the Lagrangian form (in which the change of momentum *u* along with a particle, *du*/*dt*, is used) and the Eulerian form. With the former, advection term does not explicitly appear. Later in this course, we will also discuss semi-Lagrangian method for solving the transport problems. In this chapter, we discuss only the Eulerian advection equation.

3.2. Linear convection – 1-D wave equation

3.2.1. The wave equations

The classical 2nd-order hyperbolic wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial u^2}{\partial x^2}. \tag{2}$$

The equation describes wave propagation at a speed of c in two directions.

The 1st-order equation that has properties similar to (2) is

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0. \tag{3}$$

Note that Eq.(2) can be obtained from Eq.(3), by taking a time derivative of (3) and resubstituting (3) into the new equation.

For a pure initial value problem with initial condition

$$u(x, 0) = F(x), \quad -\infty < x < \infty,$$

the exact solution to (3) is $u(x,t) = F(x\text{-}ct)$, which we have obtained earlier using the method of characteristics. We know that the solution represents a signal propagating at speed c.

3.2.2. Centered in time and space (CTCS) FD scheme for 1-D wave equation

We apply the centered in time and space (CTCS) scheme to Eq.(2):

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} - c^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = 0 . \tag{4}$$

We find for this scheme,

$$\tau = O(\Delta t^2 + \Delta x^2) .$$

Performing von Neumann stability analysis, we can obtain a quadratic equation for amplification factor λ:

$$\lambda_{\pm} = 1 - 2\mu^2 \sin^2\left(\frac{k\Delta x}{2}\right) \pm 2\mu \sin\left(\frac{k\Delta x}{2}\right)\left[\mu^2 \sin^2\left(\frac{k\Delta x}{2}\right) - 1\right]^{1/2}$$

where $\mu = \frac{c\Delta t}{\Delta x}$ which is the fraction of zone distance moved in Δt at speed c.

Let $\theta = \mu \sin\left(\frac{k\Delta x}{2}\right)$, we have

$$\lambda_{\pm} = 1 - 2\theta^2 \pm 2\theta[\theta^2 - 1]^{1/2} .$$

We want to see under what condition, if any, $|\lambda_{\pm}| \le 1$.

We consider two possible cases.

<u>Case I:</u> If $\theta \le 1$, then λ is complex:

$$\lambda_{\pm} = 1 - 2\theta^2 \pm i2\theta[1-\theta^2]^{1/2}$$

$$\rightarrow \qquad |\lambda_{\pm}|^2 = (1-2\theta^2)^2 + 4\theta^2[1-\theta^2] = 1$$

Therefore, when $\theta \le 1$, the amplification factor is always 1, which is what we want for pure advection!

$$\theta \le 1 \rightarrow \ p^2 \sin^2\left(\frac{k\Delta x}{2}\right) \le 1$$

We want the above to be true for all k, therefore $p^2 \le 1$ has to be satisfied for all value of $\sin^2()$.

$$p^2 \le 1 \ \rightarrow \ p = \frac{c\Delta t}{\Delta x} \le 1,$$

which is the same as the condition we obtained earlier using energy method for FTUS scheme.

<u>Case II:</u>

If $\theta \ge 1$, λ is real:

$$\lambda_{\pm} = 1 - 2\theta^2 \pm 2\theta[\theta^2 - 1]^{1/2} \rightarrow$$

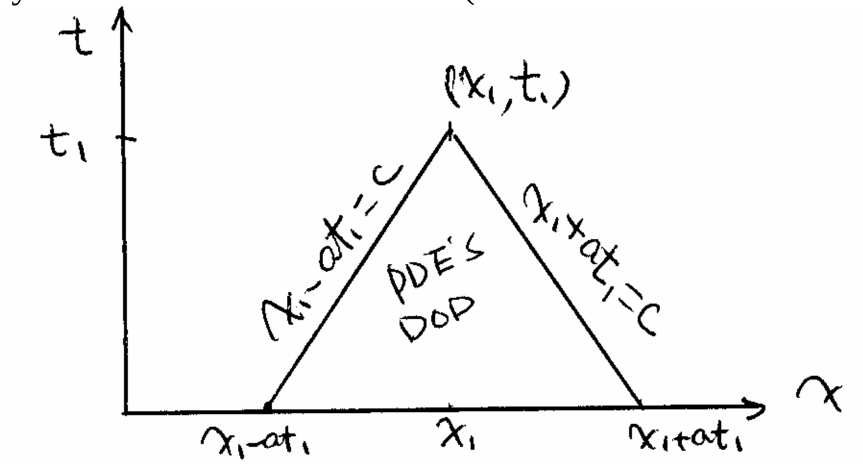
$$|\lambda_{\pm}|^2 = (1 - 2\theta^2 \pm 2\theta[\theta^2 - 1]^{1/2})^2,$$

you can show for yourself that $|\lambda_{\pm}| > 1$ therefore the scheme is unstable.

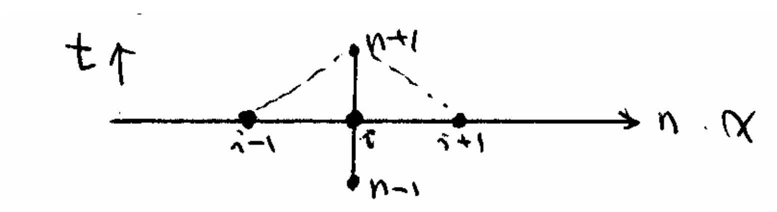
3.2.3. Courant-Friedrichs-Lewy (CFL) Stability Criterion

Let's consider the stability condition obtained above using the concept of <u>domain of dependence.</u>

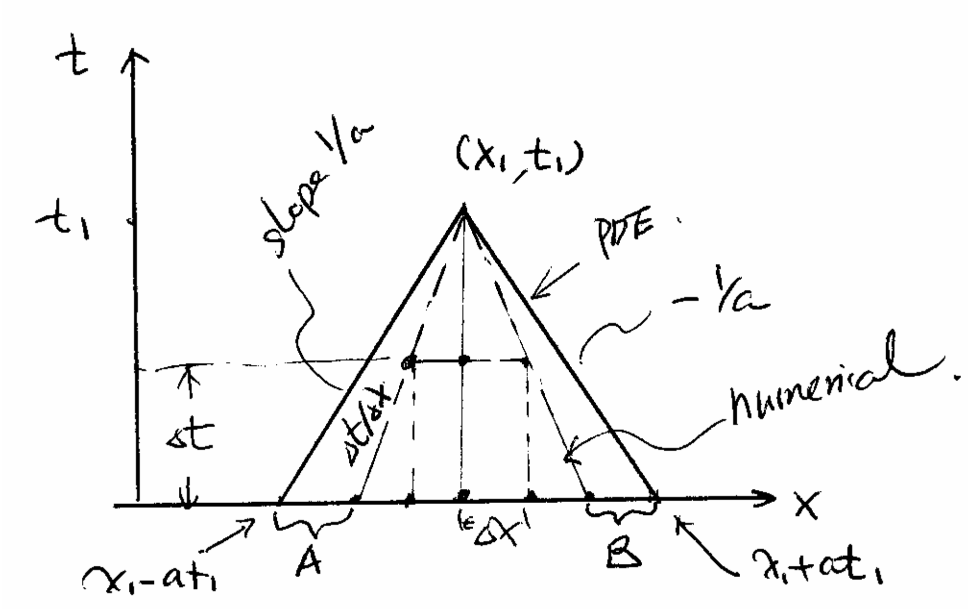
Recall from earlier discussion, the solution at (x_1, t_1) depends on data in the interval $[x_1 - at_1, x_1 + at_1]$, and the D.O.D. is the area enclosed by the two characteristics lines (note here a instead of c is the advection speed)



Based on the following discretization stencil,



we can construct a <u>numerical domain of dependence</u> below:



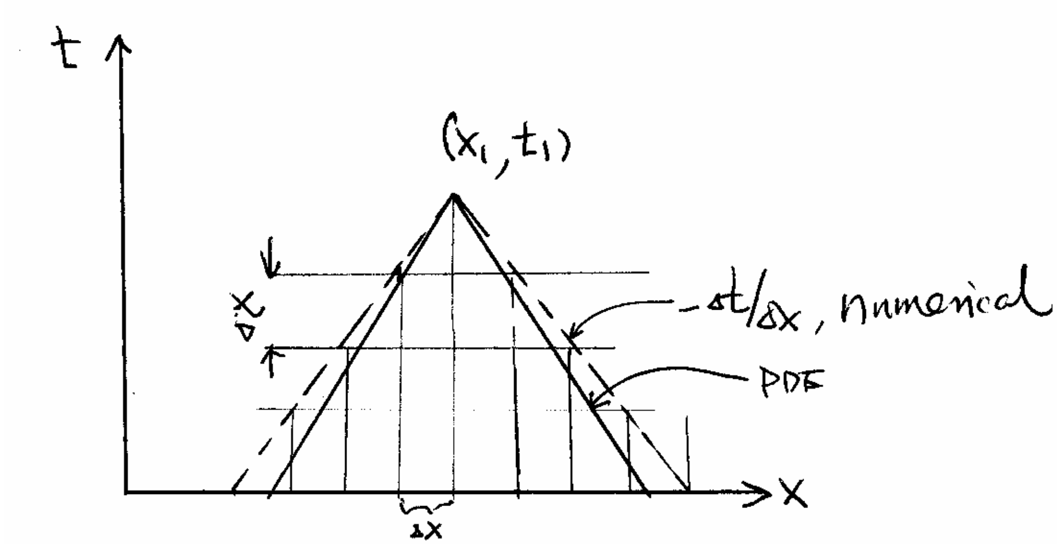
Case I: When the numerical DOD is smaller than the PDE's DOD (which usually happens when Δt is large), the numerical solution cannot be expected to converge to the true solution, because the numerical solution is not using part of the initial condition, e.g., the initial values in the intervals of A and B. The true solution, however, is definitely dependent on the initial values in these intervals. Different initial values there will result in different true solutions, while the numerical solution remains unaffected by their values. We therefore cannot expect the solutions to match.

The numerical solution must then be unstable. Otherwise, the Lax's Equivalence theorem is violated.

The above situation occurs when $\Delta t / \Delta x > 1/c$ $\rightarrow$ unstable solution. This agrees with the result of our stability analysis.

Case II: When $\Delta t / \Delta x = 1/c$, the PDE DOD coincides with the numerical DOD, the scheme is stable.

Case III: When Δt / Δx < $1/c$, the PDE DOD is contained within the numerical DOD:



the numerical solution now fully depends on the initial condition. It is possible for the scheme to be stable. In the case of CTCS scheme, it is indeed the case.

Definition: $\dfrac{c\Delta t}{\Delta x} = \sigma =$ <u>Courant number</u>

The condition that $\sigma \le 1$ for stability is known as the <u>Courant-Friedrichs-Lewy (CFL)</u> stability criterion.

<u>The CFL condition requires that the numerical domain of dependence of a finite difference scheme include the domain of dependence of the associated partial differential equation.</u>

<u>Satisfaction of the CFL condition is a necessary, not a sufficient condition for stability</u>.

E.g., the second-order centered-in-time and fourth-order centered-in-space scheme for a 1-D advection equation requires $\sigma \le 0.728$ for stability whereas the D.O.D condition requires that $\sigma \le 2$.

Example 2: The forward-in-time, centered-in-space scheme is absolutely unstable, even if the CFL condition is satisfied.

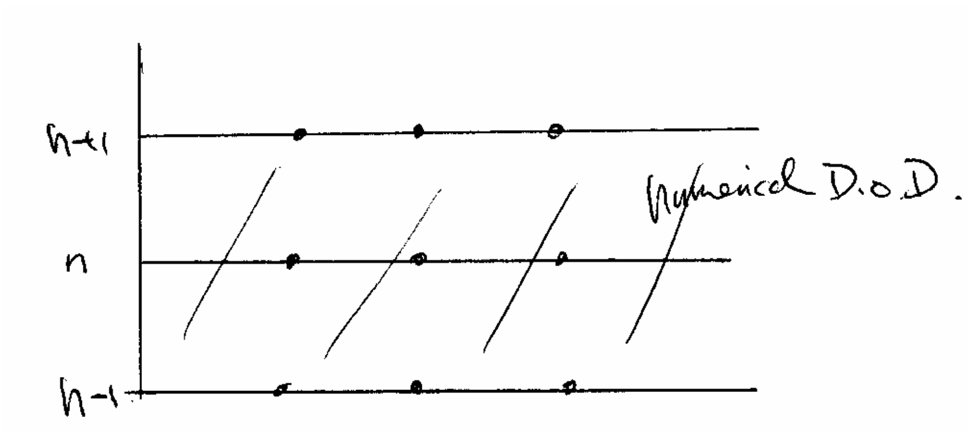
The DOD concept explains why implicit schemes can be unconditionally stable – it is because their numerical DOD always contains the PDE's DOD

e.g., the second-order in time and space implicit scheme for wave equation (2):

$$\delta_{tt} u^n = \frac{c^2}{4}[\delta_{xx} u^{n+1} + 2\delta_{xx} u^n + \delta_{xx} u^{n-1}].$$

is stable for all σ.

The numerical DOD is:



The numerical DOD covers the PDE's DOD.

Read Durran sections 2.2.2 and 2.2.3, which discuss the CFL criterion using the forward-in-time upstream-in-space (also called upwind) scheme.