25. Show that a problem (1)—(4) with more complicated boundary conditions, say, $u(0, t) = 0$, $u(L, t) = h(t)$, can be reduced to a problem for a new function v satisfying conditions $v(0, t) = v(L, t) = 0$, $v(x, 0) = f_1(x)$, $v_t(x, 0) = g_1(x)$ but a nonhomogeneous wave equation. *Hint.* Set $u = v + w$ and determine w suitably.

11.4 D'Alembert's Solution of the Wave Equation

It is interesting to note that the solution (17), Sec. 11.3, of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad c^2 = \frac{T}{\rho} \tag{1}$$

can be immediately obtained by transforming (1) in a suitable way, namely, by introducing the new independent variables[1]

$$v = x + ct, \qquad z = x - ct. \tag{2}$$

Then u becomes a function of v and z, and the derivatives in (1) can be expressed in terms of derivatives with respect to v and z by the use of the chain rule in Sec. 8.7. Denoting partial derivatives by subscripts, we see from (2) that $v_x = 1$ and $z_x = 1$. For simplicity let us denote $u(x, t)$, as a function of v and z, by the same letter u. Then

$$u_x = u_v v_x + u_z z_x = u_v + u_z.$$

Applying the chain rule to the right side and using $v_x = 1$ and $z_x = 1$ we find

$$u_{xx} = (u_v + u_z)_x = (u_v + u_z)_v v_x + (u_v + u_z)_z z_x = u_{vv} + 2u_{vz} + u_{zz}.$$

We transform the other derivative in (1) by the same procedure, finding

$$u_{tt} = c^2(u_{vv} - 2u_{vz} + u_{zz}).$$

By inserting these two results in (1) we obtain (cf. footnote 1 in Appendix 3.1)

$$\boxed{u_{vz} \equiv \frac{\partial^2 u}{\partial z\, \partial v} = 0.} \tag{3}$$

Obviously, the point of the present approach is that the resulting equation

[1]We mention that the general theory of partial differential equations provides a systematic way for finding this transformation which will simplify the equation. Cf. Ref. [C14] in Appendix 1.

Our result shows that the two initial conditions and the boundary conditions determine the solution uniquely.

The solution of the wave equation by the Laplace transformation and the Fourier transformation will be shown in Secs. 11.13 and 11.14.

In the next section we turn to the **heat equation,** which is another partial differential equation of basic importance.

Problems for Sec. 11.4

Using (6), sketch a figure (of the type of Fig. 270 in Sec. 11.3) of the deflection $u(x, t)$ of a vibrating string (length $L = 1$, ends fixed) starting with initial velocity zero and the following initial deflection $f(x)$, where k is small, say, $k = 0.01$.

1. $f(x) = k \sin \pi x$
2. $f(x) = kx(1 - x)$
3. $f(x) = k(x - x^3)$
4. $f(x) = k(x^2 - x^4)$
5. $f(x) = k(1 - \cos 2\pi x)$
6. $f(x) = k \sin^2 \pi x$

Using the indicated transformations, solve the following equations.

7. $u_{xy} = u_{xx}$ $(v = y, z = x + y)$
8. $yu_{xy} = xu_{xx} + u_x$ $(v = y, z = xy)$
9. $u_{xx} - 2u_{xy} + u_{yy} = 0$ $(v = y, z = x + y)$
10. $u_{xx} = u_{yy}$ $(v = y + x, z = y - x)$
11. $u_{yy} + u_{xy} - 2u_{xx} = 0$ $(v = x + y, z = 2y - x)$
12. $u_{xx} - 4u_{xy} + 3u_{yy} = 0$ $(v = x + y, z = 3x + y)$

Types and normal forms of linear partial differential equations. An equation of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y) \tag{8}$$

is said to be **elliptic** if $AC - B^2 > 0$, **parabolic** if $AC - B^2 = 0$, and **hyperbolic** if $AC - B^2 < 0$. (Here A, B, C may be functions of x and y, and the type of (8) may be different in different parts of the xy-plane.)

13. Show that

    Laplace's equation $u_{xx} + u_{yy} = 0$ is elliptic,
    the **heat equation** $u_t = c^2u_{xx}$ is parabolic,
    the **wave equation** $u_{tt} = c^2u_{xx}$ is hyperbolic,
    the **Tricomi equation** $yu_{xx} + u_{yy} = 0$ is of mixed type (elliptic in the upper half-plane and hyperbolic in the lower half-plane).

14. If the equation (8) is *hyperbolic*, it can be transformed to the *normal form* $u_{vz} = F^*(v, z, u, u_v, u_z)$ by setting $v = \Phi(x, y)$, $z = \Psi(x, y)$, where $\Phi = const$ and $\Psi = const$ are the solutions $y = y(x)$ of the equation $Ay'^2 - 2By' + C = 0$ (cf. Ref. [C12]). Show that in the case of the wave equation (1),

$$\Phi = x + ct, \qquad \Psi = x - ct.$$

15. If (8) is *parabolic*, the substitution $v = x$, $z = \Psi(x, y)$, with Ψ defined as in Prob. 14, reduces it to the *normal form* $u_{vv} = F^*(v, z, u, u_v, u_z)$. Verify this result for the equation $u_{xx} + 2u_{xy} + u_{yy} = 0$.
16. **(Airy equation)** Show that by separating variables we can obtain from the **Tricomi equation** the Airy equation $G'' - yG = 0$. (For solutions, see p. 446 of Ref. [1] listed in Appendix 1. See also Review Prob. 30 for Chap. 4.)

EXAMPLE 1. Vibrating string if the initial deflection is triangular

Find the solution of the wave equation (1) corresponding to the triangular initial deflection

$$f(x) = \begin{cases} \dfrac{2k}{L}x & \text{if } 0 < x < \dfrac{L}{2} \\ \dfrac{2k}{L}(L - x) & \text{if } \dfrac{L}{2} < x < L \end{cases}$$

and initial velocity zero. (Fig. 270 on the next page shows $f(x) = u(x, 0)$ at the top.)

Solution. Since $g(x) \equiv 0$, we have $B_n^* = 0$ in (12), and from Example 1 in Sec. 10.5 we see that the B_n are given by (5), Sec. 10.5. Thus (12) takes the form

$$u(x, t) = \frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi}{L}x\cos\frac{\pi c}{L}t - \frac{1}{3^2}\sin\frac{3\pi}{L}x\cos\frac{3\pi c}{L}t + - \cdots\right].$$

For plotting the graph of the solution we may use $u(x, 0) = f(x)$ and the above interpretation of the two functions in the representation (17). This leads to the graph shown in Fig. 270. ▮

It is very interesting that the solution (17) can also be obtained very quickly by a suitable transformation of the wave equation, following an ingenious idea by d'Alembert, which we discuss in the next section.

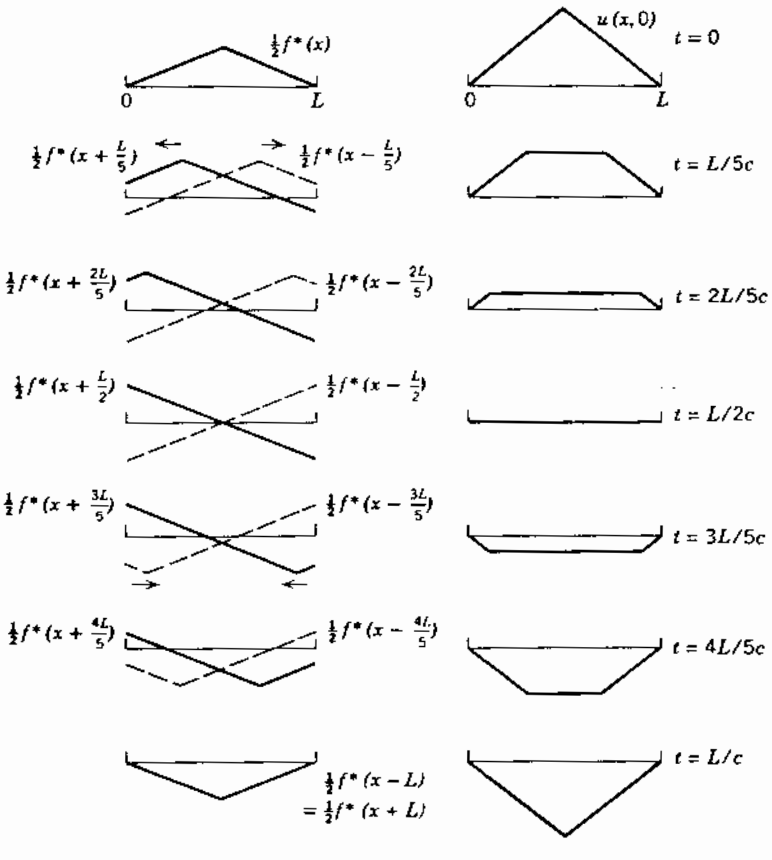

Fig. 270. Solution $u(x, t)$ in Example 1 for various values of t (right part of the figure) obtained as the superposition of a wave traveling to the right (dashed) and a wave traveling to the left (left part of the figure)