

Chapter Two. Finite Difference Methods

2.1. Introduction

2.1.1. The Concept of Finite Difference Method

In FDM, we represent continuous fluid flow problems in a discrete manner, when the fluid continuum is replaced by a mesh of discrete points. The same is true for the time variable.

FDM are the simplest of all approximations, and involve a mapping:

PDE $\xrightarrow{\text{Discretization}}$ System of algebraic equations

Calculus \rightarrow algebra

Derivative \rightarrow difference

We focus on the following:

- Properties of FDM
- Derivation via several methods
- Physical interpretation in terms of characteristics
- Application to selected problems

First, we lay down a convention for notation:

Time level - superscript n - $\rho^n \sim \rho$ at time level n

$$\Delta t = \text{time interval} = t^{n+1} - t^n.$$

Most times, we use constant Δt . Occasionally, Δt changes with time.

$n-1 \sim$ past

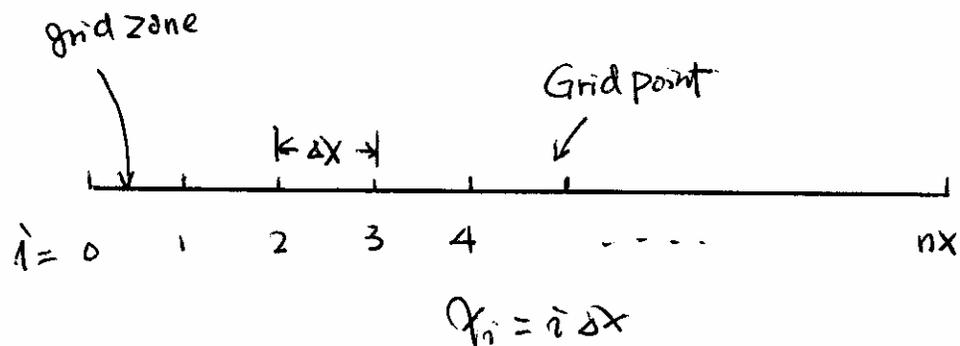
$n \sim$ present

$n+1 \sim$ future

$t = n \Delta t$ where $n =$ number of time steps $= 0, 1, 2, 3, \dots, N$

$T = N \Delta t =$ final time.

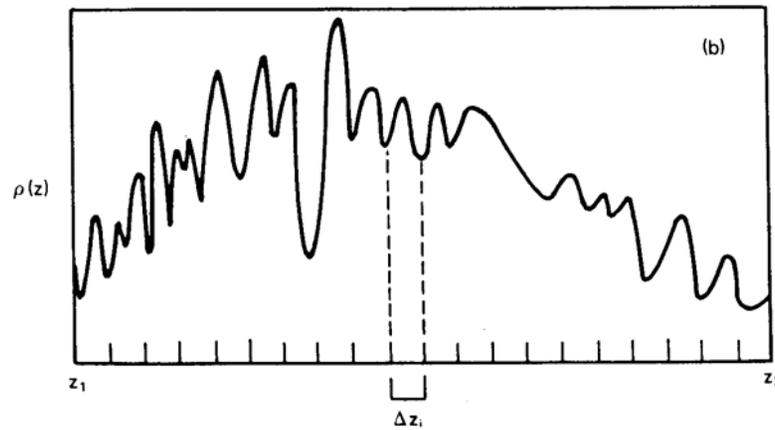
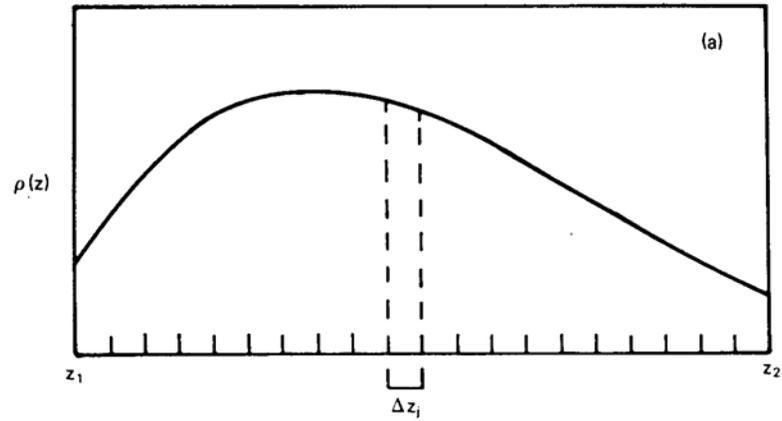
Spatial Location - subscript i, j, k , for x, y , and z .



Δx - constant grid interval

$$x_i = i \Delta x$$

Note: Discretization \rightarrow information loss – the greater the number of points, the more accurate will be the representation. See Figure.



2.1.2. Quantitative Properties of Numerical Algorithms

The governing equations (PDE's) have certain properties, and their computational counterparts should also do so.

- 1) Conservation – Typically the governing equations are written as conservation laws (which means that the integral properties over a closed volume don't change with time).

E.g., the mass conservation equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{V}).$$

If we integrate this over a closed box

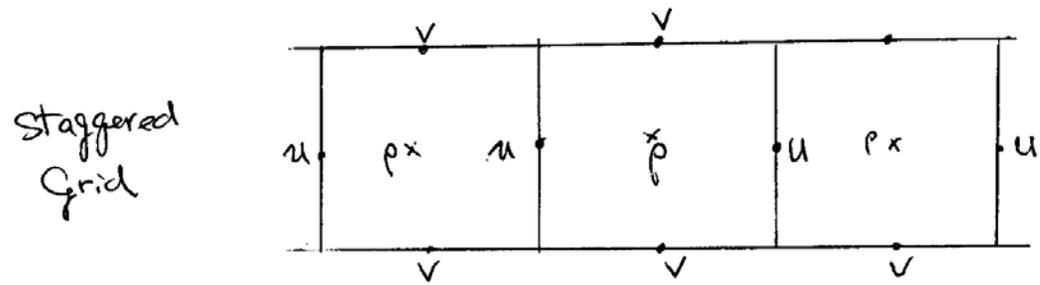
$$\frac{\partial}{\partial t} \int_{\Omega} \rho dV = - \int_{\Omega} \nabla \cdot (\rho \vec{V}) dv = 0$$

Mathematically, we can also write this as

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \vec{V} - \vec{V} \cdot \nabla \rho$$

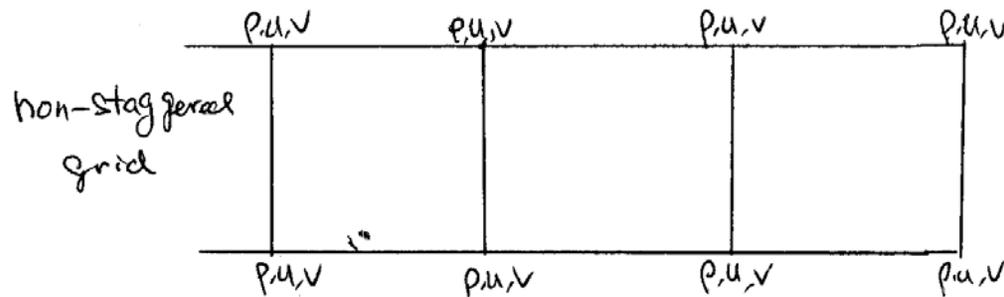
Will the numerical solution obey these rules? Not necessarily.

Consider the situation where ρ and \vec{V} are defined at separate points, ... this is how the continuity equation is really derived:



The mass within the zone changes due the mass fluxes through the sides. To get $\rho \vec{V}$ at a ρ point in this case (we are using a staggered grid), we have to average \vec{V} to ρ point, which smears out gradients!

Consider an alternative structure:



To calculate the fluxes through the sides of the grid cells shown above (which is a non-staggered Arakawa A-grid, by the way – we will come to it later), we have to perform different averaging, which result in different conservation properties of the numerical scheme.

For numerical solution to obey conservation, you must be very careful how you set up the grid, formulate the terms in the FD form, and solve the equations!!

- 2) Positivity – Physically positive quantities (mass, energy, water vapor) cannot become negative. This is not guaranteed with numerical solutions, however. Care must be taken to prevent negative values from being generated. Schemes that do so are called positive-definite schemes. A more general type is the monotonic schemes that also ensure positive definiteness, because they cannot generate new extrema that are not found in the original field.
- 3). Reversibility – Says that the equations are invariant under the transform $t \rightarrow -t$. This is important for pure transport problems, but clearly not appropriate for diffusion problems. Reversibility is actually hard to achieve even for simple advection/transportation due to unavoidable numerical errors.
- 4). Accuracy – Accuracy generally involves Computer precision, Spatial or temporal resolution, and algorithm robustness, etc.

Some of the most accurate schemes don't satisfy the above properties!!