An Example of Non-Linear Computational Instability

By NORMAN A. PHILLIPS

Massachusetts Institute of Technology, Cambridge, Massachusetts

Abstract

A particular example is constructed to demonstrate that the finite-difference solution of the non-conservative barotropic vorticity equation may have instabilities of a different nature than those obtained by other means used in the linear barotropic model. This instability arises because the grid system introduces wave lengths different than about 1 grid intervals, when such wave lengths are formed by the non-linear interaction of longer waves, the grid system introduces wave lengths in the linear waves. The apparently successful use of a smoothing process to eliminate this difficulty is discussed.

1. Introduction

Suppose we are applying the barotropic non-conservative vorticity equation to a two-dimensional flow of an ideal fluid contained in a channel between parallel walls located at \( y = 0 \) and \( y = W \), using the finite-difference methods now employed in numerical weather prediction. To make matters simple, let us restrict the initial flow patterns to those which have both the streamfunction \( \psi \) and vorticity \( z \) identically zero on both lateral boundaries (\( y = 0 \) and \( y = W \)), and so patterns which are periodic in \( y \), so that \( \psi(z, y, 0) = 0 \), \( \psi(z, y, W) = 0 \). If it is then clear that these boundary conditions will be valid for all time, and that the flow will maintain its periodic character in \( y \).

We introduce an ideal finite-difference grid, \( x = jh, j = 0, 1, 2, \ldots, J - 1 \) (even), \( y = kh, k = 0, 1, 2, \ldots, K - 1 \) (odd) where \( h \) is the space increment and \( \Delta t \) the time increment, and we suppose that \( h \) and \( K \) are such that \( L = Jh \) and \( W = Kk \).

The vorticity equation is

\[
\frac{\partial z}{\partial t} = J \frac{\partial}{\partial x}\left( \frac{\partial \psi}{\partial y} \right)
\]

(1)

(We are not here concerned with the variation of the Coriolis parameter.)

The finite-difference analogue of this which would normally be used is

\[
\nabla^2 \psi + \psi_{j+1} - \psi_j, \Delta t = \frac{\partial}{\partial x} \left( \psi_{j+1} - \psi_j \right)
\]

(2)

Here \( \psi \) and \( \psi_{j+1} \) are the usual simple centered difference approximations in the \( x \) and \( y \) directions:

\[
\psi_{j+1} = \psi_j + \frac{\Delta x}{2} \psi_y
\]

(3)

\[
\psi_y = \frac{\psi_{j+1} - \psi_{j-1}}{2 \Delta y}
\]

(4)

(2) would be applied at the interior points \( j = 0, 1, \ldots, J - 1 \) and \( k = 1, 2, \ldots, K - 1 \). At the boundary points where \( k = 0 \) or \( W \) and \( \psi_y \) are both taken to be identically zero for all time. At the points for which \( j = 0 \) and \( j = J - 1 \), the cyclic condition that \( \psi(x, k) = \psi(x, k + \Delta x) \) would be used.

The streamfunction field defined in this manner at the grid points \( j,k \) can then be represented by the finite sum:

\[
\psi_{jk} = \sum_{j=-K}^{K} \sum_{k=-K}^{K} \left[ \psi_{j+1} \cos \frac{2\pi j}{2K} + \psi_{j-1} \sin \frac{2\pi j}{2K} \right] e^{i(2\pi j x + 2\pi k y) / \lambda}
\]

(5)
We may investigate the magnitude of the unsteadiness by the same methods as those used in the previous Section 3. To find the effect of the disturbance, we consider the general circulation of the atmosphere. The ordinary instantaneously-averaged analysis of computational stability would have told us to choose a time step of 60 minutes, which would have produced a minimal increase in the disturbance. However, this analysis is not adequate, as shown by the following numerical experiment.

For a given time step of 60 minutes, the disturbance was increased by a factor of 1.2, and the resulting energy growth was measured. The results are shown in Fig. 1. As can be seen, the disturbance grows exponentially with time, and the energy growth is quadratic with time.

3. Elimination of the Instability by Smoothing

Several years ago, the writer applied the techniques of numerical prediction to the study of the general circulation of the atmosphere (Phillips, 1954). The results were obtained by making a forecast for an extended period with a 2-level geostrophic model. The equations included a term representing the effect of friction, and were applied to a simplified geostrophic model of the atmosphere. The results were then used to compute the resulting energy growth.

In an attempt to exploit this type of computational error, a similar set of equations was solved with a smoother horizontal grid interval of 20 km. The results were then compared with those obtained in the earlier experiment. Although the results were not identical, the general agreement was good, and the results were used to improve the prediction of the effect of the disturbance.

The graph of $\sqrt{t}$ is shown in Fig. 1, and the computed energy growth is shown in Fig. 2. The results are consistent with the theoretical prediction.

The smooth forecast of the energy growth is shown in Fig. 3, and the unsmoothed forecast in Fig. 4. The results are consistent with the theoretical prediction.

The results of this study have important implications for the prediction of atmospheric circulation. The technique of smoothing can be used to improve the accuracy of the forecast, and the results can be used to improve the theoretical model.

503
the sudden break-down in that forecast at around 56 days.) This suggests that these geostrophic equations do not readily transmit energy to horizontal wave lengths shorter than 700 km—
a result already familiar from the analysis by Figaro (Farjott, 1953) — since otherwise the smoothing process would have taken a noticeable amount of energy out of the system. However, the discussion above of the non-linear instability mechanism, and the success of the smoothing procedure, together indicate that even this small rate of energy transfer may be sufficient to activate non-linear computational instabilities in wave lengths shorter than 4 grid intervals if these components are not artificially removed.

In conclusion it may be appropriate to point out that misrepresentation errors similar to (5) will be encountered in solving the non-linear "balance equation" by finite differences (Bokun, 1955; Charney, 1955). This has already been noted by Shuman, who has developed some useful approximations to the straightforward but time consuming Fourier smoothing (Shuman, 1957).

Acknowledgements

This research was sponsored by the Office of Naval Research and the Geophysics Research Directorate of the Air Force under contract No. 1841(13). The numerical computations were performed at the MCT Computation Center, Cambridge, Massachusetts.

REFERENCES

Bokun, R., 1955: Numerical forecasting with the baroe

cpse model. Tellus, 7, 27—49.

Charney, J., 1955: The use of the primitive equations at


Farjott, R., 1953: On the changes in the spectral
distribution of kinetic energy for one-dimensional, non divergent flow. Tellus, 5, 225—239.

Phillips, N. A., 1956: The general circulation of the

atmosphere: a numerical experiment. Q. J. Roy.

Meteor. Soc., 82, 123—134.

Shuman, F. G., 1957: Numerical methods in weather


Res., 85, 357—361.

(Note manuscript received May 22, 1958)
where the coefficients $a_{mn}$ and $b_{mn}$ are functions of $x$. In this formulation, we take $b_{mn} = -a_{mn} = 0$, so that there are $J(K - 1) + 1$ degrees of freedom in the grid point values of $y_{i}$ and also in the coefficients $a_{mn}$ and $b_{mn}$. We see from this representation that the smallest wave length in $x$ recognized by the grid system is for $l = J/2$ and corresponds to a wave length of $2.5$. In $y$, the smallest wave length is for $m = K - 1$, and corresponds to a wave length of $2.3(K - 1)$.

Equation (2) is non-linear. If we consider the interaction of 2 components $y_{i}$ and $y_{j}$, which are characterized by the wave numbers $(l_{i}, m_{i})$ and $l_{j}$), $m_{j}$), it can be seen from (2) that they will contribute to the time rate of change of the 4 components with wave numbers $(l_{i} + l_{j}, m_{i} + m_{j})$, $(l_{i} - l_{j}, m_{i} - m_{j})$, $(l_{i} - l_{j}, m_{i} + m_{j})$, and $(l_{i} - l_{j}, m_{i} - m_{j})$. This non-linear interaction determines the transfer of kinetic energy between different parts of the spectrum in this type of flow, and, in the meteorological problem, becomes very important when forecasts are to be made for any extended period of time.

We now recall that any distribution of $y$ in the grid network $b_{mn}$ can be resolved into the Fourier sum $A_{mn}$, containing only wave numbers $l = 0, 1, \ldots, J/2$ and $m = 1, 2, \ldots, K$. It is then clear that the interaction of $y_{i}$ and $y_{j}$ with each other will not be interpreted correctly when $l_{i} + l_{j} > J/2$ and/or when $m_{i} + m_{j} > K$. For example, if $l_{i} + l_{j} = J/2$, with $l_{i} > J/2$, we would find the following type of misrepresentation to occur:

$$
\begin{align*}
\cos \frac{2\pi l_{i}}{J} (l_{i} + l_{j}) &= \cos \frac{2\pi l_{j}}{J} (J - r) - \cos \frac{2\pi l_{j}}{J} r, \\
\sin \frac{2\pi l_{i}}{J} (l_{i} + l_{j}) &= \sin \frac{2\pi l_{j}}{J} (J - r) - \sin \frac{2\pi l_{j}}{J} r.
\end{align*}
$$

Thus, instead of affecting wave number $J - r$, the components $y_{i}$ and $y_{j}$ will affect wave number $r$. A similar misrepresentation will occur in the $m$ wave numbers whenever $m_{i} + m_{j} > K - 1$.

2. An example of instability from this source

The potential seriousness of this misrepresentation can be seen by constructing an artificial example. We take only 2 components:

$$
\begin{align*}
y_{1} &= C_{l} \cos \frac{\pi l}{J} + S_{l} \sin \frac{\pi l}{J} \\
y_{2} &= U_{l} \cos \frac{\pi l}{J} \\
\end{align*}
$$

Thus $l_{i} = J/4$, $m_{i} = 2K/3$, and $l_{j} = J/2$, $m_{j} = K/6$. The misrepresentation which occurs is of the form

$$
\begin{align*}
l_{i} &= J/4, \\
m_{i} &= 2K/3, \\
l_{j} &= J/2, \\
m_{j} &= K/6.
\end{align*}
$$

Since $l_{i} = l_{j}$ in this case is equal to $l_{i}$, and $m_{i} = m_{j}$ is equal to zero, no new harmonics are generated by the finite-difference interaction of $y_{i}$ and $y_{j}$. The correct finite-difference solution of this particular example is then described by the three ordinary non-linear difference equations:

$$
\begin{align*}
C_{l_{i} - 1} - C_{l_{i}} + t \frac{\partial C_{l_{i}}}{\partial t} &= 0, \\
S_{l_{i} - 1} - S_{l_{i}} + t \frac{\partial S_{l_{i}}}{\partial t} &= 0, \\
U_{l_{i} - 1} - U_{l_{i}} &= 0.
\end{align*}
$$

These are the result of inserting (6) into (2) and (3). Although $m_{i}$-linear, they are simple enough to be solved. We first find that $U_{l_{i}}$ has the constant value $x$ for even $r$ and the constant value $B$ for odd $r$. $C_{l_{i}}$ or $S_{l_{i}}$ then satisfies the difference equation:

$$
C_{l_{i} - 2} - 2 C_{l_{i}} + C_{l_{i} + 2} = 0,
$$

where $\cos \theta - 1 = \frac{1}{2} \sin 2\theta$ is a constant. This difference equation has four solutions:

$$
\begin{align*}
e^{\zeta_{1}} &\quad (\gamma_{1})^{\zeta_{1}} e^{\zeta_{1}} &\quad (\gamma_{1})^{\zeta_{1}} e^{\zeta_{1}} \\
(\gamma_{2})^{\zeta_{2}} &\quad (\gamma_{2})^{\zeta_{2}} e^{\zeta_{2}} &\quad (\gamma_{2})^{\zeta_{2}} e^{\zeta_{2}}
\end{align*}
$$

If $\gamma$ and $\beta$ have the same sign, $\theta$ is a real number, and two of the solutions will amplify exponentially. This "instability" cannot be eliminated by reducing $\Delta t$.

If $A$ and $B$ are of opposite sign, but small enough in magnitude so that $1 + \frac{1}{2} \sin 2\theta > 1$, $\gamma$ is pure imaginary and we have four neutral solutions. However, if $A$ and $B$ are of opposite sign but large enough in magnitude so that $1 + \frac{1}{2} \sin 2\theta < 1$, the solutions are again of the form $\exp \pm \gamma \sqrt{2} t$ where $\cos \theta = 1 + \frac{1}{2} \sin 2\theta$. These again will amplify with time, since $\theta$ will be real. Thus, when $A$ and $B$ are of opposite sign, the instability can be eliminated by reducing $\Delta t$. 

502